Alternative Formulas for Measuring Industry Contributions to Labor Productivity Change

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Draft of May 28, 2014

Although the effects on wages and living standards of productivity growth tend to be broad-based, the productivity gains themselves are often concentrated in a handful of industries. An analysis of industry contributions to aggregate productivity growth rate is therefore an important part of understanding an economy's productivity performance.

Much of the literature on productivity focuses on the role of total factor productivity (TFP; also known as multifactor productivity) as part of a framework that accounts for an economy's growth based on quantity indexes of inputs of labor, capital, human capital and land. In the appendix we discuss a formula for decomposing TFP into industry contributions based on Domar's (1961) weighting scheme. Under certain assumptions, the sum of the products of the industries' Domar weights and the industries' TFP growth exactly equals the economy's aggregate TFP growth. This result is noteworthy because in most frameworks for analyzing productivity growth, the movement of inputs between industries with different productivity levels causes changes in aggregate productivity that cannot be explained using a formula that looks only at within-industry productivity growth.

Another strand of the productivity literature focuses on labor productivity, and that is the main topic of the present paper. Labor productivity compares real output growth to a simple measure of the quantity of labor employed calculated as the sum of hours worked or FTEs (full-time equivalent employees). Physical and intangible capital can be made to grow faster by investing more, and human capital can be increased by education and training, but the amount of labor resources that is available to an economy is for the most part predetermined (though some changes may be brought about by immigration or emigration.) For an economy that does not have large flows of cross-border investment income, the standard of living and wages ultimately depend on labor productivity.

In this paper I first review three of the existing formulas for decomposing the change in an economy's aggregate labor productivity into contributions from industries and then develop some new formulas. The industry contributions to productivity change are supposed to add up exactly to the change in aggregate labor productivity that has been calculated by comparing the growth rate of aggregate real output (real GDP in the case of the total economy, or real business sector output if non-market producers are excluded) to the growth of labor.

I. The Two Challenges in Designing a Decomposition Formula for Productivity

Besides differing in labor productivity growth rates, industries can differ in labor productivity levels because of factors such as differences in inputs of physical capital, intangible capital and human capital and differences in technology.[1] Differences in labor productivity levels between industries mean that the movement of labor between industries a role in aggregate productivity growth, which complicates the analysis of industry sources of productivity growth.[2] Treating the change in aggregate productivity that arises from labor reallocation as an effect that cannot be assigned to individual industries is one option, but this solution is not entirely satisfactory because productivity analysts prefer to have a set of industry contributions that completely account for aggregate productivity growth. In this paper we therefore assume that the goal is to have a complete decomposition into industry contributions that has an appealing interpretation. Nevertheless, a certain amount of arbitrariness is inherent in any assignment of a jointly produced effect to individual contributors, so choosing the assignment scheme with appealing properties can be viewed as a difficult problem.

Researchers who want to calculate industry contributions to productivity change for Canada, the United States, and certain other countries also face a second challenge. This challenge arises because, unlike the Laspeyres ("constant price") volume measures that were once used to calculate the official measures of real output, the Fisher index formulas now used generate non-additive volume measures. The modern formulas use direct Fisher indexes to measure growth in real output over adjacent periods (such as the current year and the preceding year), and chained Fisher indexes for comparisons over longer intervals of time. Fisher or chained index measures of the real output of the components of an aggregate will generally fail to sum up to the directly calculated Fisher or chained volume measure of the output of the aggregate.

Consequently, even though summing the nominal value added of every industry yields nominal GDP, a discrepancy will generally exist between the sum of the Fisher index measures of the real value added of every industry and real GDP as calculated from the Fisher formula. This discrepancy between the sum of the parts and the whole can easily translate into a discrepancy between the sum of industry contributions to productivity change and the aggregate that they seek to decompose. In particular, even if no labor is reallocated to or from any industry, a discrepancy may arise between the weighted average of the within-industry productivity changes and the directly calculated measure of economy-wide productivity.

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[1] Differences in productivity levels related to technology may represent temporary disequilibria caused by factors such as lags in responses to changes in technology, trade opportunities and other changes in circumstances, slow mobility of capital and labor, or barriers to competition.

[2] Edward Denison identified the movement of labor from low productivity agriculture to high productivity manufacturing as a source of US labor productivity growth in the first half of the Twentieth Century, so Nordhaus (2002) calls the contribution to growth from labor reallocation "the Denison effect."

This paper presents some alternative ways to handle the labor reallocation effect and some new ways to handle the problem of non-additivity of the direct Fisher and chained Fisher indexes. In particular, researchers who are interested in certain types of questions have had to give up on exact additivity and accept errors of approximation. This paper develops formulas that are suitable for answering those questions and that have virtually no error of approximation. Nevertheless, although the new measures developed in this paper have important advantages, different questions require different answers, and no single solution is the correct one for all purposes.

II. Some Approaches from the Literature on Industry Contributions to Aggregate Labor Productivity

The "Traditional" Decomposition and the CSLS Decomposition

For many years statistical agencies used Laspeyres quantity indexes (and Paasche price indexes) to calculate measures of real output, also known as measures of output volume. This framework holds prices constant, so the volume measures are additive. With additive volume measures, deriving additive decompositions of the change in aggregate labor productivity is straightforward.

Let Z_{it} be the measure of labor productivity in the arbitrary industry i that is based on Laspeyres quantity indexes that use period 0 prices and the Z_t be the corresponding measure of aggregate labor productivity. Then:

$$Z_t = \sum_i l_{it} Z_{it} \tag{1}$$

A simple formula for decomposing aggregate productivity change into industry contributions was called "the traditional decomposition" by de Avillez (2012) and Dumagan (2013) because of its long history. The traditional decomposition is:

$$\begin{aligned}(Z_t - Z_0)/Z_0 &= \sum_i [l_{it}Z_{it} - l_{i0}Z_{i0}]/Z_0 \\ &= \sum_i [l_{i0}(Z_{it} - Z_{i0}) + Z_{i0}(l_{it} - l_{i0}) + (Z_{it} - Z_{i0})(l_{it} - l_{i0})]/Z_0 \\ &= \sum_i [(l_{i0} Z_{i0}/Z_0)(Z_{it}/Z_{i0} - 1) + (Z_{i0}/Z_0)(l_{it} - l_{i0}) + ((Z_{it} - Z_{i0})/Z_0)(l_{it} - l_{i0})] \qquad (2)\end{aligned}$$

The direct contribution from within-industry productivity growth in the arbitrary industry i is given be the first term on the right hand side of equation (2). The remaining terms in equation (2) are known as reallocation effects because they reflect the changes in industry shares of employment. The middle term is the static reallocation effect and the last terms is the dynamic reallocation effect.

If period 0 is the base period for measuring prices then $l_{i0} Z_{i0}/Z_0$ in the first term can be simplified to w_{i0} and the direct contribution, denoted by $\hat{c}_i^D$, can be written as $w_{i0} g(Z_i)$, where $g(Z_i)$ is defined as $Z_{it}/Z_{i0}-1$. Substituting w_{i0} for $l_{i0} Z_{i0}/Z_0$ in the other terms of equation (2) yields a convenient expression for calculating the traditional decomposition:

$$(Z_t - Z_0)/Z_0 = \sum_i w_{i0}\, g(Z_i) + w_{i0}(l_{it}/l_{i0} - 1) + w_{i0}\, g(Z_i)(l_{it}/l_{i0} - 1) \tag{3}$$

A decomposition that de Avillez (2012) terms "the CSLS decomposition" improves on the traditional decomposition by modifying the reallocation terms so that they have a useful economic interpretation. The CSLS reallocation terms take into account the comparative productivity level of the industry that is receiving or releasing labor resources because the impact of the movement of labor on the economy's aggregate productivity depends on whether productivity is higher in the industries where it is redeployed than in the industries that it exited.

Many papers on industry contributions to productivity growth have used formulas that tend to show a fast-growing industry's reallocation contribution as negative if that industry has a below average productivity level and a shrinking industry's reallocation contribution as positive if that industry has a below average productivity level.[3] An industry that releases labor can be viewed as placing that labor in a pool where it is available to any industries, and an industry that absorbs more labor can be viewed as depleting the pool that is available to all industries. Using the economy's average level of productivity as a benchmark for measuring reallocation effects is therefore reasonable.

In the CSLS decomposition, the measures of productivity level and growth in the second and third terms of equation (2) are expressed as deviations from means. In the second term in equation (2), we can substitute $Z_{i0} - Z_0$ for Z_{i0} because $\sum_i (l_{it} - l_{i0}) = 0$. Similarly, in the third term of equation (2), the change in the weighted averages of the productivity levels can be subtracted from $Z_{it} - Z_{i0}$. If the total contribution of industry i to aggregate productivity growth is $\hat{c}_i^D + \hat{c}_i^R$, the combined static and dynamic labor reallocation effect $\hat{c}_i^R$ is:

$$\begin{aligned}\sum_i \hat{c}_i^R &= \sum_i \left[(Z_{i0} - Z_0)/Z_0 + ((Z_{it} - Z_{i0}) - (Z_t - Z_0))/Z_0\right](l_{it} - l_{i0}) \\ &= \sum_i \left[(w_{i0} - l_{i0}) + (w_{i0}(Z_{it}/Z_{i0} - 1) - l_{i0}(Z_t/Z_0 - 1))\right](l_{it}/l_{i0} - 1) \\ &= \sum_i \left[w_{i0} - l_{i0} + w_{i0}\, g(Z_i) - l_{i0}\, g(Z)\right](l_{it}/l_{i0} - 1) \end{aligned} \tag{4}$$

[3] The assumption behind these formulas is that an industry's own average productivity Z_{i0} is a good proxy for the productivity of the marginal labor that it releases or absorbs. Many earlier papers on industry contributions to productivity growth use formulas that make a fast-growing industry's reallocation contribution negative if that industry has a below average productivity level and a shrinking industry's reallocation contribution positive if that industry has a relatively low productivity level. See Basu and Fernald (1995), Nordhaus (2002), Stiroh (2002), CSLS () and Reinsdorf and Yuskavage (2010).

In equation (4) an industry that takes labor from other industries has a positive static contribution to aggregate productivity if its productivity level is above average. It has a positive dynamic contribution to aggregate productivity if its productivity growth is above average. Also, an industry that releases labor to be employed by other industries has a positive static contribution to aggregate productivity if its productivity level is below average, and it has a positive dynamic contribution to aggregate productivity if its productivity growth is below average.

The GEAD Decomposition of Labor Productivity Change

Statistical agencies such as Statistics Canada and the Bureau of Economic Analysis no longer publish the Laspeyres volume measures that make the traditional and CSLS decompositions exactly additive. Researchers have therefore sought a new formula that is suitable for use with the Fisher index volume measures that are now published. In particular, researchers have sought a way to decompose a measure of aggregate labor productivity based on a Fisher or chained Fisher volume index into industry contributions that add up to the right total.

A formula for industry contributions to productivity growth that solves the additivity problem created by the use of Fisher indexes was developed by Tang and Wang (2004). Because the formula is applicable to superlative quantity index measures, to chained measures and to Laspeyres volume measures, Dumagan (2013) terms it the "generalized exactly additive decomposition" or GEAD.

The GEAD formula normalizes the prices of individual industries by dividing each individual price by the deflator used for the top level aggregate. Let P_{it} denote the price index for the value added of industry i, let F_t be the aggregate price index at time t, and let $p_{it} = P_{it}/F_t$. Similarly, in the base period 0, $p_{i0} = P_{i0}/F_0$. In addition, let labor productivity in industry i be $X_{it} = (Y_{it}/P_{it})/L_{it}$, where L_{it} is a simple sum of hours over all labor types. Finally let $l_{it} = L_{it}/L_t$, the share of aggregate labor inputs used by industry i. Then if Y_i is aggregate nominal value added in period t, aggregate labor productivity in period t equals $X_t = (Y_i/F_t)/L_t$, or,

$$X_t = \sum_i p_{it}\, l_{it}\, X_{it} \tag{5}$$

The change in labor productivity from period 0 to period t is:

$$g(X) \equiv \frac{X_t - X_0}{X_0}$$

$$= \frac{\sum_i p_{it}\, l_{it}\, X_{it}}{\sum_i p_{i0}\, l_{i0}\, X_{i0}} - 1 \tag{6}$$

Now let $w_{i0} = Y_{it}/Y_t$ and note that $w_{i0} = p_{i0}\, l_{i0}\, X_{i0} \,/ \sum_j p_{j0}\, l_{j0}\, X_{j0}$. Then

$$g(X) = \sum_i w_{i0}\,[(p_{it}/p_{i0})(l_{it}/l_{i0})(X_{it}/X_{i0}) - 1]$$

$$= \sum_i w_{i0}\,[(p_{it}/p_{i0})(l_{it}/l_{i0})(\,1+g(X_i)) - 1]$$

$$= \sum_i w_{i0}\,[(p_{it}/p_{i0})(l_{it}/l_{i0}) - 1)(\,1+g(X_i)) + g(X_i)]$$

$$= \sum_i\ (w_{i0}/l_{i0})[(p_{it}/p_{i0})l_{it} - l_{i0}](1+g(X_i)) + \sum_i w_{i0}\,g(X_i) \qquad (7)$$

The final term in equation (7) is the direct contribution from within-industry labor productivity growth. For the arbitrary industry i this contribution is $c_i^D = w_{i0}\,g(X_i)$.

Tang and Wang (2004) break the first term on the right side of equation (7) up into two pieces, which I will call the static reallocation effect and the dynamic reallocation effect. Defining x_{i0} as X_{i0}/X_0, the relative productivity of industry i in period 0, the total contribution from reallocation, denoted by c_i^R can be written as:

$$c_i^R = (w_{i0}/l_{i0})[(p_{it}/p_{i0})l_{it} - l_{i0}](1+g(X_i))$$

$$= (w_{i0}/p_{i0}l_{i0})(p_{it}l_{it} - p_{i0}l_{i0}) + (w_{i0}/p_{i0}l_{i0})(p_{it}l_{it} - p_{i0}l_{i0})g(X_i)$$

$$= x_{i0}(p_{it}l_{it} - p_{i0}l_{i0}) + x_{i0}(p_{it}l_{it} - p_{i0}l_{i0})g(X_i) \qquad (8)$$

Reallocation of labor towards industry i means that $l_{it} > l_{i0}$. Assuming that $g(X_i) > -1$, the reallocation contribution will be positive if $l_{it}/l_{i0} > p_{i0}/p_{it}$ If $p_{it} = p_{i0}$ then c_i^R is positive when the arbitrary industry i is a net recipient of reallocated labor and negative when it releases labor to be reallocated to other industries.

III. Why seek a New Alternative?

The GEAD formula is versatile enough to produce exactly additive contributions regardless of the type of index that the statistical agency has used to create the output volume measures. Nevertheless, it has two features that are undesirable for some purposes. First, counting above average growth in the labor employed by an industry as automatically contributing in a positive way to aggregate productivity seems to lack economic meaning. On the assumption that industries with high average labor productivity as measured by x_{i0} (or x_{it}) also have high marginal labor productivity, to show that an industry with a low level of productivity that takes labor from a representative average industry is making a negative contribution to aggregate productivity, the formula for the reallocation contribution could be modified by replacing x_{i0} with $x_{i0} - 1$. Substitution of $x_{i0} - 1$ for x_{i0} in of the formula for c_i^R would, however, necessitate some further adjustments so that the contributions continue to add up to the correct total; $\sum_i (p_{it} l_{it} - p_{i0} l_{i0}) + \sum_i (p_{it} l_{it} - p_{i0} l_{i0}) g(X_i)$ may not equal 0.

A second feature of the formula for c_i^R is the direct way that a price change can affect an industry's contribution to aggregate productivity growth. A sufficiently large increase in the prices of the goods that an industry produces (or fall in the prices of the goods that it uses as intermediate inputs) will cause p_{it}/p_{i0} to exceed l_{i0}/l_{it} , making $c_i^R > 0$. For example, if there is a disruption in the foreign supply of a product that competes with a domestic industry, the domestic industry may enjoy a price windfall that will count as part of its contribution to the economy-wide productivity growth.

Counting a pricing windfall as contributing to productivity growth is inconsistent with the abstract definition of productivity growth as an outward movement in the production possibility frontier caused by improvements in technology or the organization of production. It is also inconsistent with the practical definition of productivity growth as a change in the output quantity index in excess of the growth in the input quantity index.

The difference between the way that an output price increase affects c_i^R and the way that a price increase affects a superlative quantity index such as a Fisher or Törnqvist index is worth clarifying. In a superlative quantity index, other things being equal, a price rise makes the affected good's average share weight larger, thereby making the index more sensitive to a substitution of that good for other goods. If the quantity of every good has the same growth rate, so that there is no substitution, a rise in a price will have no effect on the quantity index.

In many of the models used in the productivity literature on TFP, productivity gains are inversely related to price changes because the cost savings from above average productivity growth are passed on to buyers. Thus from the perspective of the TFP literature, including price changes in a contribution to productivity change formula is likely to have the effect of canceling out some of the productivity gains that are included in the contribution term for within-industry productivity growth. Conversely, sub-par productivity growth may tend to cause above average price increases that result in positive reallocation effects in the GEAD framework.

Diewert (2013, p. 5)) has recently developed a formula that isolates the price effect in the GEAD decomposition by breaking apart the p_{it} and l_{it} parts of the $p_{it} l_{it}$ term in the original GEAD formula. This three factor version of the GEAD opens up a route for excluding the real price change effect from the reallocation effect. The sum over all industries of Diewert's price change effect term is usually small in magnitude, so researchers who do not want to include the direct effects of price changes in an industry's productivity contribution can simply exclude the price change term from the total contribution of each industry. After the price change effects are excluded the decomposition will no longer be exactly additive, but the error of approximation should not be large. However, below we derive formulas for Fisher measures of change in labor product that are exactly additive and that have more transparent economic interpretations that the Diewert (2013) decomposition.

An Illustration of the Formula Differences using the Business Sector of Canada

To illustrate the differences between the existing decomposition methods, table 1 presents decompositions of productivity growth in Canada across two-digit NAICS sectors between 2000 and 2010 calculated from data from Statistics Canada's Canadian Productivity Accounts.[4] The calculations test the CSLS decomposition, the GEAD formula of Tang and Wang (2004), and the three-factor version of the GEAD decomposition of Diewert (2013).

The static and dynamic reallocation effect contributions of the CSLS and GEAD formulas are not shown in table 1 for reasons of space, but their total can be inferred by subtracting the within-industry effect from the overall total effect. Industry total contributions based on the three factor GEAD are also omitted from the table because they are exactly the same as the ones from the GEAD formula of Tang and Wang (2004). For the three factor decomposition, table 1 shows total industry contributions net of the price change contribution term. These net total contributions might be used by a researchers who wants to exclude direct price effects from the measure of contributions to productivity change.

All three methods decompose the compound annual aggregate labor productivity growth rate of 0.8 percent that Canada experienced between 2000 and 2010. The CSLS decomposition, which attributes aggregate productivity growth to productivity growth within each sector and to level and growth effects of reallocation of labor across sectors implies that 75 percent of Canada's aggregate productivity growth is the result of within-industry productivity improvements. The GEAD estimates are even more emphatic than the CSLS estimates that improvements in labor productivity within sector are the primary driver of aggregate productivity growth in Canada. According to the GEAD formula, 103 per cent of aggregate productivity growth can be accounted for by within-sector productivity gains because the static and dynamic labor reallocation effects are offsetting. The Diewert (2013) decomposition finds that 81 per cent of the growth is the result of rising labor productivity and 21 per cent comes from labor reallocation. As anticipated, the overall sum of the real price change contributions is small.

The large differences between the methods start to become apparent when we ask which sectors are generating Canada's productivity growth. The CSLS decomposition implies that the most important sectors for aggregate labor productivity growth over the period were agriculture, forestry, fishing and hunting (16.2% of the total growth), manufacturing (18.3%), wholesale trade (24.6%), retail trade (16.7%), and finance, insurance, real estate, rental, and leasing (FIRE) (17.2%). With the exception of FIRE, the contributions to aggregate productivity from these

[4] I am grateful to Andrew Sharpe and Matthew Calver for these calculations.

sectors come mostly from within sector productivity. The CSLS method also identifies three sectors with substantial negative effects on Canada's productivity growth: mining and oil and gas extraction (-7.6%), construction (-6.8%), and administration and support, waste management and remediation services (ASWMRS) (-5.7%).

The GEAD, on the other hand, finds that mining and oil and gas extraction and construction are the biggest *positive* contributors, raising Canada's labor productivity growth rate by 0.28 percentage points (35.4%) and 0.44 percentage points (54.7%). This method also identifies three sectors as having negative contributions to aggregate productivity growth: agriculture, forestry, fishing and hunting (-6.9%), utilities (-0.7%), and manufacturing (-104%). Again, the contrast with the CSLS decomposition could scarcely be sharper. However, in the case of FIRE, the GEAD and CSLS contributions are closer to agreement, as the FIRE sector has a sizeable positive reallocation effect using either method.

Why do the two types of decomposition disagree so sharply about which sectors are driving aggregate labor productivity growth in Canada? The problem is not the contributions of within sector labor productivity; the GEAD gives a generally similar pattern to the CSLS decomposition of within sector contributions, albeit not as similar as the Diewert (2013) decomposition. Instead, the main source of disagreement seems to be the impact of price changes on the reallocation terms of the GEAD. The role of price changes is evident from the Diewert's (2013) three factor decomposition (though the direct effect of prices shown in table 1 may understate their impact because prices changes also enter into several covariance terms in the Diewert decomposition.) In this decomposition, growth in real output prices in the mining and oil and gas extraction, construction and manufacturing accounts for 36.9%, 20.8% and -41.4% of aggregate productivity growth respectively.

In a decomposition of the change in nominal business sector value added over the same time period, these three sectors again stand out because mining and oil and gas extraction and construction generate the biggest contributions to nominal aggregate output growth, while manufacturing is the only sector that negatively affected aggregate growth (see Table 2). Indeed, the pattern of contributions to nominal output growth closely corresponds to the pattern of contributions to productivity growth from the GEAD. The GEAD formula is general enough to work with chained measures of real output growth because it gives an exactly additive decomposition regardless of how the aggregate deflators F_0 and F_t are specified. This means that it can be used to decompose the change in *nominal* output per hour of labor input by specifying $F_0 = F_t = 1$. Thus, one way to think of the GEAD formula is as a renormalized decomposition of nominal output per hour.

In the case of the mining and oil and gas extraction industry, a substantial part of the difference between CSLS and GEAD arises from a large increase in output prices. In the Diewert (2013) decomposition, the contribution for this industry excluding the price change effect is –0.01 percentage points annually, not far from the total contribution using the CSLS formula.

In the case of the manufacturing sector, the CSLS decomposition suggests that labor reallocation had very little effect even though the share of labor allocated to the manufacturing sector decreased greatly, falling from 20 percent to 14 percent from 2000 to 2010. One might expect such a large amount of labor reallocation to generate a large impact on aggregate productivity. Recall, however, that the CSLS decomposition calculates the labor reallocation effect by considering how the industry's level and growth of productivity compare to the mean level and growth of productivity:

$$\sum_i \hat{c}_i^R = \sum_i \left[(Z_{i0}-Z_0)/Z_0 + ((Z_{it}-Z_{i0})-(Z_t-Z_0))/Z_0\right](l_{it}-l_{i0})$$

The level (45.86 chained 2007 dollars per hour) and change of productivity (4.57 chained 2007 dollars per hour) in manufacturing are close to the mean level (43.60) and change (3.60). The small sizes of the deviations from the mean imply small contributions of labor reallocation.

The GEAD and three factor decompositions treat the falling labor share in manufacturing very differently from the CSLS formula. They show large negative contributions to aggregate productivity for manufacturing. The three factor decomposition also shows that falling output prices (they fell 16 percent over the period) account for a significant part of the very large negative reallocation effect for manufacturing in the GEAD.

Table 1: Alternative Decompositions of Business Sector Labor Productivity in Canada at the Two-Digit NAICS Level, 2000-2010

	CSLS		GEAD		Three Factor GEAD (Diewert, 2013)		
	Within Sector Productivity Effect	Total with Reallocation Effect Included	Within Sector Productivity Effect	Total with Reallocation Effect Included	Within Sector Productivity Effect	Direct Effect of Change in Prices	Total excluding Direct Price Effect
Business sector industries	**0.60**	**0.80**	***0.84***	**0.80**	**0.65**	**-0.02**	**0.82**
Agriculture, forestry, fishing and hunting	0.09	0.13	0.11	-0.06	0.08	-0.05	-0.01
Mining and oil and gas extraction	-0.30	-0.06	-0.21	0.28	-0.31	0.29	-0.01
Utilities	0.00	0.01	0.00	-0.01	0.00	-0.02	0.01
Construction	0.01	-0.05	0.01	0.44	0.01	0.17	0.27
Manufacturing	0.19	0.15	0.23	-0.83	0.18	-0.33	-0.50
Wholesale trade	0.20	0.20	0.23	0.10	0.21	-0.05	0.15
Retail trade	0.15	0.13	0.17	0.14	0.17	-0.05	0.19
Transportation and warehousing	0.03	0.03	0.03	0.03	0.03	0.00	0.03
Information and cultural industries	0.09	0.09	0.10	0.06	0.10	-0.04	0.10
FIRE	0.04	0.14	0.04	0.20	0.05	-0.09	0.29
Professional, scientific and technical services	0.06	0.05	0.06	0.18	0.07	0.05	0.13
ASWMRS	0.01	-0.05	0.01	0.11	0.01	0.03	0.08
Arts, entertainment and recreation	0.00	-0.01	0.00	0.01	0.00	0.00	0.01
Accommodation and food services	0.01	0.03	0.02	0.02	0.02	0.01	0.01
Other private services	0.03	0.02	0.04	0.12	0.04	0.05	0.07

Table 2: Contributions to Nominal Business Sector Output Growth and GEAD Contributions to Business Sector Labor Productivity in Canada at the Two-Digit NAICS Level, 2000-2010

Sector	Nominal GDP, 2000, millions	Nominal GDP, 2010, millions	Contribution to Nominal Output Growth	GEAD Contribution to Productivity Growth	Relative Contribution to Nominal Output Growth	Relative Contribution to Productivity Growth
Business sector industries	**777008**	**1150015**	**48.0**	**0.80**	**100.0**	**100.0**
Construction	47727	113256	8.4	0.44	17.6	55.0
FIRE	116542	181849	8.4	0.20	17.5	25.0
Mining and oil and gas extraction	61143	114686	6.9	0.28	14.4	35.0
Professional, scientific and technical services	48657	86112	4.8	0.18	10.0	22.5
Retail trade	49230	82555	4.3	0.14	8.9	17.5
Wholesale trade	51790	81964	3.9	0.10	8.1	12.5
Other private services	38652	65775	3.5	0.12	7.3	15.0
ASWMRS	22462	42920	2.6	0.11	5.5	13.8
Transportation and warehousing	43653	63101	2.5	0.03	5.2	3.8
Information and cultural industries	31429	49447	2.3	0.06	4.8	7.5
Accommodation and food services	22219	32157	1.3	0.02	2.7	2.5
Utilities	26278	35310	1.2	-0.01	2.4	-1.3
Arts, entertainment and recreation	7087	10970	0.5	0.01	1.0	1.3
Agriculture, forestry, fishing and hunting	21244	22971	0.2	-0.06	0.5	-7.5
Manufacturing	188895	166941	-2.8	-0.83	-5.9	-103.8

Table 3: Exact and Approximate Estimates of Aggregate Growth of Labor Productivity in the US from 1998 to 2012

Percent per Year

	1999	2000	2001	2002	2003	2004	2005
Based on official contributions	2.51	1.81	0.69	2.79	2.96	2.54	1.38
Based on approximated contributions	2.48	1.73	0.61	2.80	2.96	2.49	1.40

	2006	2007	2008	2009	2010	2011	2012
Based on official contributions	0.85	0.32	-0.13	2.56	3.34	0.31	0.56
Based on approximated contributions	0.86	0.32	-0.12	2.52	3.39	0.31	0.59

Table 4: Symmetric Fisher and GEAD Decompositions of Growth of Labor Productivity in the US from 1998 to 2012

Percentage Points

Sector or Industry	Total Contribution		Within-Sector Productivity Change Contribution			Reallocation Effect Contribution		
	Symmetric and CSLS Fisher	GEAD	Sym-metric Fisher	CSLS Fisher	GEAD	Sym-metric Fisher	CSLS Fisher	GEAD
Total economy	**24.71**	**24.74**	**26.45**	**26.82**	**28.42**	**-1.75**	**-2.11**	**-3.69**
Farms, forestry, fishing	0.48	0.44	0.54	0.56	0.62	-0.06	-0.07	-0.18
Oil and gas extraction	0.68	**1.64**	0.05	0.04	0.40	**0.63**	0.64	**1.23**
Other mining	0.26	0.77	0.19	0.19	0.19	0.06	0.07	0.58
Utilities	0.28	0.31	0.48	0.49	0.55	-0.20	-0.21	-0.24
Construction	-0.63	0.28	-0.81	-0.78	-0.75	0.18	0.15	1.02
Durable goods manufacturing excluding computers	**3.13**	**-1.03**	**2.83**	**2.82**	**2.99**	0.30	0.31	**-4.01**
Computer and electronic products	**3.97**	-0.18	**4.30**	**4.40**	**4.74**	-0.33	-0.43	**-4.91**
Nondurable Manufacturing	**2.00**	1.03	1.93	**1.97**	2.04	0.07	0.03	-1.01
Wholesale & retail trade	**2.58**	1.30	**2.58**	**2.57**	**2.64**	0.01	0.01	-1.34
Transportation and warehousing	0.47	0.56	0.55	0.56	0.64	-0.08	-0.10	-0.07
Publishing and motion picture and sound recording	1.11	0.61	1.27	1.29	1.29	-0.16	-0.18	-0.69
Broadcasting, data processing, telecomm. and internet	**2.78**	0.57	**3.37**	**3.46**	**3.53**	-0.59	**-0.68**	**-2.96**
Finance	**2.72**	1.24	**2.61**	**2.62**	**2.83**	0.11	0.11	-1.58
Real estate, rental and leasing	**4.35**	**4.17**	**3.86**	**3.88**	**3.90**	0.50	0.48	0.27
Professional, scientific, and technical services	1.26	**3.55**	1.03	1.04	1.07	0.22	**0.22**	**2.48**
ASWMRS	1.00	1.21	1.08	1.10	1.10	-0.08	-0.10	0.11
Educational services	-0.43	0.58	-0.03	-0.03	-0.03	-0.40	-0.40	0.61
Health care and social assistance	**-0.96**	**3.04**	0.24	0.24	0.25	**-1.21**	**-1.21**	**2.80**
Arts, entertainment, recreation, accommodation, food services	-0.62	1.04	0.17	0.19	0.20	-0.79	-0.81	0.85
Other services, except government	-0.57	-0.03	-0.60	-0.60	-0.59	0.03	0.03	0.56
Government	0.85	**3.64**	0.81	0.82	0.83	0.04	**0.03**	**2.81**

IV. Symmetric Approaches to Measuring Contributions to Aggregate Labor Productivity

Weights that treat the Time Periods Symmetrically

Many authors assign an economic meaning to the dynamic reallocation term as a measure of the Baumol effect (see, for example, Nordhaus (2002).[5] Yet from a mathematical point of view, this term is present because of an asymmetry in the way that the GEAD and CSLS decomposition treat the two time periods. They use only period 0 as the source of the weights in the term for the contribution from within-industry productivity growth (such as the last term in equation (7)). If a version of Irving Fisher's time reversal test is applied to these decompositions, when the comparison is run backwards from period t to period 0, the pattern of industry contributions may be inconsistent with the pattern seen when the comparison is run in the normal direction.

To modify the traditional decomposition to have weights that treat both time periods symmetrically, average the labor shares of the two periods and also average the productivity levels of the two periods. Let $\bar{l}_i = (l_{it}+l_{i0})/2$ and let $\bar{Z}_i = (Z_{it}+Z_{i0})/2$. Then,

$$\begin{aligned}(Z_t - Z_0)/Z_0 &= \textstyle\sum_i [l_{it}Z_{it} - l_{i0}Z_{i0}]/Z_0 \\ &= \textstyle\sum_i [\bar{l}_i Z_{i0}[(Z_{it}-Z_{i0})/Z_{i0}] + \bar{Z}_i(l_{it}-l_{i0})]/Z_0 \\ &= \textstyle\sum_i 0.5[w_{i0}(1+l_{it}/l_{i0})g(Z_i) + ((Z_{it}/Z_t)(Z_t/Z_0) + Z_{i0}/Z_0)(l_{it}-l_{i0})] \qquad (9)\end{aligned}$$

With weights based on two-period averages, the formula for the direct contribution of industry i's productivity index is $0.5w_{i0}(1+l_{it}/l_{i0})g(Z_i)$ and the dynamic reallocation effect term vanishes.

The reallocation term of equation (9) can be rewritten using deviations from means, as is done in the reallocation effect of the CSLS decomposition. This gives the symmetrically weighted version of the CSLS reallocation effect:

$$\hat{c}_i^{R*} = (l_{it} - l_{i0})0.5[(Z_t/Z_0)(Z_{it} - Z_t)/Z_t + (Z_{i0} - Z_0)/Z_0]. \qquad (10)$$

Fisher Index Measure of Real Output with Asymmetric Weights

The Fisher index is defined as a geometric mean of a Laspeyres index and a Paasche index, but it can also be expressed as a weighted arithmetic average of a Laspeyres index and a Paasche index. Let Q_t^L, Q_t^P and Q_t^F be the top-level Laspeyres, Paasche and Fisher quantity indexes, respectively

[5] Baumol hypothesized that over time an increasing share of expenditures would go to products with stagnant productivity. As a result, in the long run, aggregate productivity growth would experience a slowdown. This effect came to be known as "Baumol's disease."

and let $\lambda = (Q_t^P)^{0.5}/[(Q_t^P)^{0.5} + (Q_t^L)^{0.5}]$ (or, equivalently, $(P_t^P)^{0.5}/[(P_t^P)^{0.5} + (P_t^L)^{0.5}]$, where P_t^P and P_t^L are the Paasche and Laspeyres price indexes.) Then:

$$Q_t^F = \frac{Q_t^L[Q_t^P]^{0.5} + Q_t^P[Q_t^L]^{0.5}}{[Q_t^P]^{0.5} + [Q_t^L]^{0.5}}$$

$$= \lambda Q_t^L + (1-\lambda) Q_t^P \tag{11}$$

By using λ to weight on Laspeyres volume measure and $1-\lambda$ to weight the Paasche volume measure, the Fisher measure of labor productivity can be written as a sum of two productivity measures that are themselves additive.

The CSLS decomposition is exactly additive when a Laspeyres volume measure is used to measure the real output used as the numerator of Z_{it}. To calculate a Laspeyres volume measure for the arbitrary industry i, nominal output in time period t is deflated by a Paasche price index, denoted by P_{it}^P. On the other hand, to calculate the Paasche volume measure for industry i, its nominal output in period 0 is multiplied by the industry's Laspeyres price index P_{it}^L and then normalized by dividing by the aggregate Laspeyres price index, P_t^L. The industry's output in period t is then deflated by P_t^L.

Let V_{i0} be the nominal output of industry i in time 0 (which is also known output in value terms). Then the Paasche volume measure of the labor productivity of industry i is:

$$z_{i0} = V_{i0}\,(P_{it}^L/P_t^L)/L_{i0}\,. \tag{12}$$

In equation (11), the industry price indexes are divided by the aggregate price index to normalize the z_{i0} to have the same weighted average as the Z_{i0} from the Laspeyres volume framework. The weighted average of the z_{i0} is aggregate labor productivity measured at the prices of period 0:

$$Z_0 = \sum_i l_{i0} z_{i0}. \tag{13}$$

The aggregate volume measure for period t based on the Laspeyres price index is:

$$z_t = \sum_i (V_{it}/P_t^L)/\sum_i L_{it}$$

$$= \sum_i l_{it} z_{it} \tag{14}$$

where $z_{it} = (V_{it}/P_t^L)/L_{it}$. Using the fact that $V_{i0}P_{it}^L / \sum_j V_{j0}P_{jt}^L = w_{i0}P_{it}^L/P_t^L = (l_{i0}z_{i0}/Z_0)(P_{it}^L/P_t^L)$, we have:

$$(z_t - Z_0)/Z_0 = \sum_i [l_{i0}(z_{it} - z_{i0}) + z_{i0}(l_{it} - l_{i0}) + (z_{it} - z_{i0})(l_{it} - l_{i0})]/Z_0$$

$$= \sum_i [(l_{i0} z_{i0}/Z_0)(z_{it}/z_{i0} - 1) + (z_{i0}/Z_0)(l_{it} - l_{i0}) + ((z_{it} - z_{i0})/Z_0)(l_{it} - l_{i0})]$$

$$= \sum_i w_{i0}(P^L_{it}/P^L_t)g(z_i) + \sum_i \left[(z_{i0} - Z_0)/Z_0 + ((z_{it} - z_{i0}) - (z_t - Z_0))/Z_0\right](l_{it} - l_{i0}) \quad (15)$$

We can therefore calculate the direct contribution from the productivity growth in industry i to the Fisher measure of aggregate productivity growth as:

$$\tilde{c}^D_i = w_{i0}[\lambda g(Z_i) + (1-\lambda)(P^L_{it}/P^L_t)g(z_i)] \quad (16)$$

The corresponding formula for the contribution of the reallocation effect to the Fisher measure of productivity change is:

$$\tilde{c}^R_i = (l_{it} - l_{i0})\big[\lambda[(Z_{i0} - Z_0)/Z_0 + ((Z_{it} - Z_{i0}) - (Z_t - Z_0))/Z_0] +$$
$$(1-\lambda)\left[(z_{i0} - Z_0)/Z_0 + ((z_{it} - z_{i0}) - (z_t - Z_0))/Z_0\right]$$

$$= (l_{it} - l_{i0})[\lambda Z_{i0} + (1-\lambda)z_{i0} - Z_0]/Z_0 +$$
$$(l_{it} - l_{i0})[\lambda[(Z_{it} - Z_{i0}) - (Z_t - Z_0)] + (1-\lambda)[(z_{it} - z_{i0}) - (z_t - Z_0)]]/Z_0 \quad (17)$$

$$= (l_{it}/l_{i0} - 1)[w_{i0}[\lambda + (1-\lambda)(P^L_{it}/P^L_t)] - l_{i0}] +$$
$$(l_{it}/l_{i0} - 1)[\lambda[w_{i0}(Z_{it}/Z_{i0} - 1) - l_{i0}(Z_t/Z_0 - 1)]$$
$$+ (1-\lambda)[w_{i0}(P^L_{it}/P^L_t)(z_{it}/z_{i0} - 1) - l_{i0}(z_t/Z_0 - 1)] \quad (18)$$

Fisher Index Measure of Real Output with Symmetric Weights

The Laspeyres volume measure with symmetric weights is shown in equation (9) above. The corresponding Paasche volume measure is:

$$(z_t - Z_0)/Z_0 = \sum_i \left[\bar{l}_i z_{i0}[(z_{it} - z_{i0})/z_{i0}] + (z_{it} + z_{i0})(l_{it} - l_{i0})\right]/Z_0$$

$$= \sum_i 0.5[w_{i0}(P^L_{it}/P^L_t)(1 + l_{it}/l_{i0})g(z_i) + \sum_i 0.5(l_{it} - l_{i0})(z_{it} + z_{i0})/Z_0$$

$$= \sum_i 0.5[w_{i0}(P^L_{it}/P^L_t)(1 + l_{it}/l_{i0})g(z_i) + \sum_i (l_{it} - l_{i0})(w_{i0}/l_{i0})(z_{it}/z_{i0} + 1)/2 \quad (19)$$

The relative productivity level of industry i compared to the average industry is measured by $(w_{i0}/l_{i0})(z_{it}/z_{i0}+1)/2$. Using the fact that $\sum_i (l_{it}-l_{i0})=0$, the last term in equation (19) can be written in deviation form as:

$$\sum_i (l_{it}-l_{i0})(w_{i0}/l_{i0})(z_{it}/z_{i0}+1)/2 = \sum_i (l_{it}-l_{i0})[(w_{i0}/l_{i0})(z_{it}/z_{i0}+1) - (Z_t/Z_0+1)]/2$$

$$= \sum_i (l_{it}/l_{i0} - 1)[w_{i0}(z_{it}/z_{i0}+1) - l_{i0}(Z_t/Z_0+1)]/2$$

Combining the Laspeyres and Paasche symmetrically weighted measures gives the symmetrically weighted Fisher contribution to growth formula. The direct contribution to aggregate Fisher productivity growth of industry i's productivity growth is:

$$\tilde{\tilde{c}}_i^D = w_{i0}[0.5(1+l_{it}/l_{i0})][\lambda g(Z_i) + (1-\lambda)(P_{it}^L/P_t^L)g(z_i)] \tag{20}$$

The symmetrically weighted version of the Fisher reallocation effect is, then:

$$\tilde{\tilde{c}}_i^R = (l_{it}-l_{i0})0.5[\lambda[(Z_t/Z_0)(Z_{it}-Z_t)/Z_t + (Z_{i0}-Z_0)/Z_0] + (1-\lambda)[(z_t/Z_0)(z_{it}-z_t)/z_t + (z_{i0}-Z_0)/Z_0]] \tag{21}$$

This expression can be simplified to:

$$\tilde{\tilde{c}}_i^R = 0.5w_{i0}(l_{it}/l_{i0}-1)\ [\lambda[Z_{it}/Z_{i0} - Z_t/Z_0] + (1-\lambda)(P_{it}^L/P_t^L)[z_{it}/z_{i0} - z_t/Z_0]]$$

$$= \sum_i 0.5w_{i0}(P_{it}^L/P_t^L)[\ (1+l_{it}/l_{i0})g(z_i) + (z_{it}/z_{i0}+1)(l_{it}/l_{i0}-1)] \tag{22}$$

V. Chained Fisher Volume Measures of Productivity Change

Besides working well with Fisher measures of productivity change comparing two years directly, the Tang and Wang (2004) decomposition is flexible enough to be applied to measures of productivity change based on chained indexes. This section shows how to achieve an additive decomposition of a chained volume measure with other kinds of decompositions.

To work with chained volume measures, add a time subscript t to the notation for the contributions to denote the contribution to the change from year t–1 to year t. Also let Q_t^F denote the direct Fisher quantity index from year t–1 to year t and let Z_t denote the index of aggregate labor productivity from year t–1 to year t. The change in aggregate productivity from year t–1 to year t is, then,

$$[Q_t^F V_{t-1}/L_t]/[V_{t-1}/L_{t-1}] - 1 = Q_t^F/[L_t/L_{t-1}] - 1$$

$$= Z_t - 1$$

$$= \sum_i \tilde{\tilde{c}}^D_{it} + \tilde{\tilde{c}}^R_{it}. \qquad (23)$$

The chained Fisher measure of aggregate productivity change from year t–1 to year t+1, equal to $Z_t Z_{t+1}$, then has a change of:

$$[Q^F_{t+1}\, Q^F_t\, V_{t-1}/L_{t+1}]/[V_{t-1}/L_{t-1}] - 1 = [Q^F_{t+1}\, Q^F_t]/[L_{t+1}/L_{t-1}] - 1$$

$$= [Q^F_t/(L_t/L_{t-1}) - 1] + [Q^F_t/(L_t/L_{t-1})][\,Q^F_{t+1}/(L_{t+1}/L_t) - 1]$$

$$= Z_t - 1 + Z_t(Z_{t+1} - 1)$$

$$= \sum_i \tilde{\tilde{c}}^D_{it} + \tilde{\tilde{c}}^R_{it} + Z_t(\tilde{\tilde{c}}^D_{i,t+1} + \tilde{\tilde{c}}^R_{i,t+1}) \qquad (24)$$

Similarly,

$$Z_t - 1 + Z_t(Z_{t+1} - 1) + Z_t Z_{t+1}(Z_{t+2} - 1) = \sum_i \tilde{\tilde{c}}^D_{it} + \tilde{\tilde{c}}^R_{it} + Z_t(\tilde{\tilde{c}}^D_{i,t+1} + \tilde{\tilde{c}}^R_{i,t+1}) + Z_t Z_{t+1}(\tilde{\tilde{c}}^D_{i,t+1} + \tilde{\tilde{c}}^R_{i,t+1}) \qquad (25)$$

If we wish to add a third link to the chain, the additive contribution for the arbitrary industry i to $Z_t Z_{t+1} Z_{t+2} - 1$ would be calculated as: $\tilde{\tilde{c}}^D_{it} + \tilde{\tilde{c}}^R_{it} + Z_t(\tilde{\tilde{c}}^D_{i,t+1} + \tilde{\tilde{c}}^R_{i,t+1}) + Z_t Z_{t+1}(\tilde{\tilde{c}}^D_{i,t+1} + \tilde{\tilde{c}}^R_{i,t+1})$. In general, additive contributions to chained volume measures are calculated by rescaling the contributions to year-over-year productivity change so that they have a common base in the initial time period, and then summing over time.

VI. Illustration of the Chained Fisher Volume Measures of Productivity Change using Data from the US Industry Accounts

Calculating Fisher Decompositions if Laspeyres and Paasche Volumes are Unavailable

In practice the statistical agencies that publish Fisher volume measures usually do not release a Laspeyres version of the industry output measures. To implement the exactly additive Fisher decompositions that are provided in this paper, a researcher would thus have to get access to an unpublished level of detail from the industry accounts. Yet fortunately, assuming that the most detailed Fisher indexes that are available are equal to both the Laspeyres index and the Paasche index and using the Laspeyres and Paasche index formulas to aggregate these components will usually yield very accurate approximations to the above formulas.

Once the aggregate Laspeyres and Paasche indexes have been estimated by assuming that the most detailed Fisher indexes that are available are equal to both the Laspeyres index and the

Paasche index and using the Laspeyres and Paasche index formulas to aggregate these detailed indexes, the aggregate Laspeyres and Paasche indexes can be used to calibrate λ. The assumptions also imply that $g(Z_i) = g(z_i)$, so the weight on industry i in the Fisher decomposition can be calculated as $w_{i0}[\lambda + (1-\lambda)(P^L_{it}/P^L_t)]$. These procedures can be expected to yield a decomposition that has very tiny errors of approximation.

Applying these procedures to data from the Annual Industry Accounts of the US Bureau of Economic Analysis results in errors of approximation that average about 0.01 percentage points over the 14 years from 1998 to 2012. Chaining the official Fisher output measures gives a cumulative productivity growth of 24.89 percentage points over these 14 years, while chaining the sums of the approximate contributions gives a cumulative productivity growth of 24.71 percentage points. The exact and approximate aggregate labor productivity figures for each year are shown in table 3. In most years the error of approximation is under 0.05 percentage points.

Illustration of Chained Versions of the Fisher Decompositions

In Table 4, data from the US Annual Industry Accounts are used to calculate chained Fisher and chained GEAD decompositions for labor productivity change over the years 1998-2012. Table 4 reports the results of chaining the Fisher version of the CSLS decomposition, which uses $w_{i0}[\lambda + (1-\lambda)(P^L_{it}/P^L_t)]$ rather than w_{i0} for the weights. It also reports the results of chaining the symmetrically weighted Fisher decomposition.

As might be expected based on the results for Canada, the GEAD formula produces large negative reallocation effects for manufacturing and the high tech industries of computer manufacturing and data processing and telecommunications services. In the cases of computers and other durable goods manufacturing the total contribution to US productivity growth is also negative. As computers are generally thought to be the most important positive driver of US productivity growth this is a strange looking result. On the other hand, the GEAD contributions suggest that government and oil and gas extraction made notable positive contributions to US productivity growth. (Government output is generally measured under the assumption that TFP growth for the sector is zero, so the positive estimate for government probably reflects a change in the composition of the government work force that eliminated many less skilled jobs.)

On the other hand, the Fisher decompositions show that computer manufacturing and manufacturing in general made large positive contributions to US productivity growth, as did the data processing and telecommunications industries.

Finally, comparing the symmetrically weight Fisher decomposition and the CSLS-like Fisher decomposition shows that they are not very different. The sectors with negative reallocation effects have estimates that are slightly closer to zero using the symmetric Fisher decomposition, which implies slightly lower estimates of the contribution of within-sector productivity growth.

VII. Conclusion

The GEAD decomposition of Tang and Wang (2004) has been widely used because it provides exactly additive decompositions of labor productivity measures based on Fisher and chained Fisher volume measures. This paper illustrates that it can produce anomalous results for industries with a rapidly changing output price. It also develops and illustrates some new decomposition formulas for measures of labor productivity that are based on Fisher or chained indexes. The versions of these measures that are feasible to implement from the published data were found to have discrepancies between the sum of the industry contributions and the directly calculated measure of aggregate productivity change that were very small, on the order of a few hundredths of a percentage point. Furthermore, the Fisher decomposition formulas produce estimates of sector and industry contributions to aggregate productive growth with plausible, useful economic interpretations.

Appendix

Industry Contributions to Total Factor Productivity in a Growth Accounting Framework

There are many ways to measure the relative distance between production possibility frontiers (PPFs) attributable to growth of TFP, but two of them are especially relevant. Let Y^F be the vector of final outputs, M^M be the vector of imported intermediate inputs, and let L_t, K_t and N_t be the economy's endowments of primary factors of labor, capital and natural resources (land) at time t.[6] For purposes of exposition, it is convenient to assume that the economy is at a profit-maximizing point on its production possibility frontier, which rules out most kinds of disequilibria, and that aggregate technology exhibits constant returns to scale. Define the revenue function $R_t(L,K,N;P^F, P^M)$ as the function that gives the maximum value of revenue $P^F\cdot Y^F - P^M\cdot M^M$ achievable at prices (P^F, P^M) with the technology of period t and primary factor inputs (L,K,N). Then a measure of aggregate TFP based on final period prices and final period inputs is:

$$TFP^{\text{Allen-Paashe}} = R_t(L_t,K_t,N_t;P^F_t, P^M_t) / R_0(L_t,K_t,N_t;P^F_t, P^M_t)$$

$$= [R_t(L_t,K_t,N_t;P^F_t, P^M_t) / R_0(L_0,K_0,N_0;P^F_t, P^M_t)] / [R_0(L_t,K_t,N_t;P^F_t, P^M_t)/R_0(L_0,K_0,N_0;P^F_t, P^M_t)]$$

The Paasche quantity index of GDP provides an upper bound estimate of the total change in output:

$$(P^F_t\cdot Y^F_t - P^M_t\cdot M^M_t)/(P^F_t\cdot Y^F_0 - P^M_t\cdot M^M_0) \approx R_t(L_t,K_t,N_t;P^F_t, P^M_t) / R_0(L_0,K_0,N_0;P^F_t, P^M_t)$$

If technology change has the same proportional effect on output when inputs are (L_t,K_t,N_t) as when they are (L_0,K_0,N_0) then:

$$R_t(L_t,K_t,N_t;P^F_t, P^M_t)/R_t(L_0,K_0,N_0;P^F_t, P^M_t) = R_0(L_t,K_t,N_t;P^F_t, P^M_t)/R_0(L_0,K_0,N_0;P^F_t, P^M_t)$$

Furthermore, if factors of production are paid their marginal revenue product, a Paasche quantity index of inputs will provide an lower bound approximation to $R_t(L_t,K_t,N_t;P^F_t, P^M_t)/R_t(L_0,K_0,N_0;P^F_t, P^M_t)$. The Paasche quantity index of output divided by the Paasche quantity index of inputs is than an upper bound measure of the theoretical change in total factor productivity given by $TFP^{\text{Allen-Paashe}}$.

A symmetric analysis shows that under certain assumptions a Laspeyres quantity index of

$$TFP^{\text{Allen-Laspeyres}} = R_t(L_0,K_0,N_0;P^F_0, P^M_0) / R_0(L_0,K_0,N_0;P^F_0, P^M_0)$$

$$= [R_t(L_t,K_t,N_t;P^F_0, P^M_0) / R_0(L_0,K_0,N_0;P^F_0, P^M_0)]/[R_t(L_t,K_t,N_t;P^F_0, P^M_0) / R_t(L_0,K_0,N_0;P^F_0, P^M_0)]$$

[6] The time subscript on N could reflect exhaustion or new discoveries of mineral resources, or changes in the amount of land usable for agriculture caused by global warming.

$$= [R_t(L_t,K_t,N_t;P_0^F, P_0^M) / R_0(L_0,K_0,N_0;P_0^F, P_0^M)]/[R_0(L_t,K_t,N_t;P_0^F, P_0^M) / R_0(L_0,K_0,N_0;P_0^F, P_0^M)]$$

The Laspeyres quantity index of output is $(P_0^F \cdot Y_t^F - P_0^M \cdot M_t^M)/(P_0^F \cdot Y_0^F - P_0^M \cdot M_0^M)$. Under certain assumptions, dividing this index by a Laspeyres quantity index for inputs gives a theoretical lower bound for the conceptual measure given by $TFP^{Allen\text{-}Laspeyres}$. The Fisher index of TFP then has an appealing property as an average of upper and lower bounds for theoretical indexes.

Aggregate and Industry Level TFP in the Framework of the Domar Decomposition

Besides the final goods and services included in Y^F, industries also produce outputs that are used as intermediate inputs by themselves or by other industries. Assuming, for simplicity, that there are no taxes on products or tariffs, nominal GDP can be calculated as the sum of the value added of every industry. The Laspeyres (Paasche) volume measure of real GDP can also be calculated as the sum of industries' value added measured at initial (final) period prices.

The assumption that the economy is operating at a profit maximizing point on the PPF implies that at the margin reallocating inputs from one industry to another will not change the value of the revenue function. Hulten (1978) showed that in this framework the log change in aggregate TFP defined as an outward shift in the PPF can be calculated as a weighted sum of the log change in TFP of industries using the weights introduced by Domar (1961). The Domar weights add up to more than 1. Define Y_{i0} as the nominal gross output of industry i excluding intermediate inputs used within industry i and define G_0 as the total value added of all industries. Then industry i's Domar weight w_i^D equals Y_{i0} divided by aggregate value added G_0.

Let π^G be the Paasche index that measures period t prices relative to period 0 prices for G. Then the change in the Laspeyres quantity index for aggregate output, denoted $g^L(G)$, is:

$$g^L(G) = \frac{G_t/\ \pi^G - G_0}{G_0}$$

To define the aggregate quantity index of primary inputs used in the Domar decomposition I_t we must either treat detailed inputs used by different industries as different items, or assume that detailed inputs receive the same wage (or returns) everywhere they are employed. (In inputs in different industries are treated as different items in the quantity index of aggregate inputs, when labor is reallocated from a low wage industry to a high wage industry, the weight on the increase in labor in the high wage industry will be greater than the weight on the decrease in labor in the low wage industry and the aggregate input quantity index will rise.) In addition, it is assumed that that an industry's revenues from sales of output are all used to acquire intermediate inputs or pay factors of production. Thus, if J_{i0} denotes the cost of the intermediate inputs that industry i obtains from other industries plus the cost of the primary inputs employed in industry i, $J_{i0} = Y_{i0}$. Let the Laspeyres measure of the growth rate of aggregate primary inputs I be $g^L(I) = (I_t/\pi^I - I_0)/I_0$, where

π^I is a Paasche price index for inputs. Then the Laspeyres quantity index measure of aggregate TFP is

$$\begin{aligned} TFP^{Laspeyres} &= \frac{g^L(G) - g^L(I)}{1 + g^L(I)} \\ &\approx g^L(G) - g^L(I) \\ &= \sum_i w_i^D\, [g^L(Y_i) - g^L(J_i)] \\ &= \sum_i w_i^D\, TFP_i^{Laspyres} \qquad \text{(A1)} \end{aligned}$$

In the framework of the Domar decomposition, an industry's own TFP growth times its Domar weight gives its contribution to aggregate TFP growth.

Index of Labor Inputs that uses Compensation to Weight Industry-Occupation Cells

In a competitive neo-classical equilibrium, the marginal revenue product of a labor input is equal to the amount that the employer has to pay in compensation costs (wage plus benefits and social contributions) to employ the labor. However if labor is treated as a homogeneous input, the formula for the contribution of labor reallocation to aggregate productivity growth must assume that the marginal product of labor varies in direct proportion to its average product as measured by the ratio of real value added to the quantity of labor inputs used. Furthermore, differences in pay levels across industry-occupation cells may reflects differences in training, aptitude and experience. If so, industry-occupation cells should be treated as different kinds of inputs. When this is done, the role of reallocation effects (which are a kind of residual that cannot be explained by within-industry productivity growth) may be reduced.

To calculate a Laspeyres quantity index for labor inputs, let B_{it} be the nominal wage bill in year t (for convenience, we use "wages" as equivalent to compensation costs). Also, let W_t^P be the aggregate Paasche price index for wages and let W_{it}^P be the Paasche index of wages. The Laspeyres volume of labor inputs at time t is, then,

$$\begin{aligned} \hat{B}_t &= \sum_i B_{it}/W_{it}^P \\ &= \sum_i \hat{B}_{it} \\ &= B_t/W_t^P \qquad \text{(A2)} \end{aligned}$$

Let $\hat{b}_{it} = b_{it}(W_t^P/W_{it}^P) = (B_{it}/W_{it}^P)/(B_t/W_t^P)$, the share of the aggregate wage bill paid by industry i if the wage rates of year 0 had prevailed in year t, and let $b_{i0} = B_{i0}/B_0$, the industry i's share of the aggregate wage bill in year 0. Also, let $\hat{Z}_{it} = (V_{it}/P_{it}^P)/(B_{it}/W_{it}^P)$, the Laspeyres volume measure of labor input productivity. Letting $V_t = \sum_i V_{it}$ be nominal GDP at time t and $\hat{V}_t = V_t/P_t^P$, the change in the aggregate measure of Laspeyres labor input productivity is:

$$\frac{\hat{V}_t/\hat{B}_t}{V_0/B_0} - 1 = (\hat{Z}_t - \hat{Z}_0)/\hat{Z}_0$$

$$= \sum_i \left[\hat{b}_{it}\hat{Z}_{it} - b_{i0}\hat{Z}_{i0}\right]/\hat{Z}_0$$

$$= \sum_i \left[0.5(b_{i0} + \hat{b}_{it})(\hat{Z}_{it} - \hat{Z}_{i0}) + 0.5(\hat{Z}_{i0} + \hat{Z}_{it})(\hat{b}_{it} - b_{i0})\right]/\hat{Z}_0$$

$$= \sum_i 0.5(1 + \hat{b}_{it}/b_{i0})g(\hat{Z}_i) + \sum_i \left[0.5(\hat{Z}_{i0} + \hat{Z}_{it})/\hat{Z}_0\right](\hat{b}_{it} - b_{i0})$$

$$= \sum_i 0.5(1 + \hat{b}_{it}/b_{i0})g(\hat{Z}_i) + \sum_i \left[0.5(\hat{Z}_{i0} + \hat{Z}_{it} - (\hat{Z}_0 + \hat{Z}_t))/\hat{Z}_0\right](\hat{b}_{it} - b_{i0}) \qquad \text{(A3)}$$

The contribution to aggregate Laspeyres labor input productivity growth from within-industry labor input productivity growth in industry i is:

$$c_i^{L\text{-}D} = 0.5(1 + \hat{b}_{it}/b_{i0})g(\hat{Z}_i) \qquad \text{(A4)}$$

The contribution of reallocation of labor inputs to or from industry i to aggregate Laspeyres labor input productivity growth is therefore:

$$c_i^{L\text{-}R} = \left[0.5(\hat{Z}_{i0} + \hat{Z}_{it} - (\hat{Z}_0 + \hat{Z}_t))/\hat{Z}_0\right](\hat{b}_{it} - b_{i0}) \qquad \text{(A5)}$$

To derive the Paasche volume index of labor inputs, let W_{it}^L be the Laspeyres index of wages in industry i, and let W_t^L be the aggregate Laspeyres index of wages. Also, let $\hat{b}_{i0}$ be the share of the aggregate wage bill that would have been paid by industry i had the prices of period t prevailed in period 0:

$$\hat{b}_{i0} = b_{i0}(W_{it}^L/W_t^L) \qquad \text{(A6)}$$

Then the labor inputs productivity level of industry i in period 0 measured at prices of period t is:

$$\hat{z}_{i0} = \frac{V_{i0}(P_{it}^L/P_t^L)}{B_{i0}(W_{it}^L/W_t^L)} \qquad \text{(A7)}$$

and the aggregate Paasche volume productivity equals:

$$\hat{z}_0 = V_0/B_0$$

$$= \sum_i \hat{b}_{i0}\hat{z}_{i0} \tag{A8}$$

Now let $\hat{z}_{it} = (V_{it}/B_{it})(W_t^L/P_t^L)$, a normalized ratio of value added to total wages in industry i, and let $\hat{z}_t = (V_t/B_t)(W_t^L/P_t^L)$ denote aggregate labor input productivity in period t. If A_0 is the aggregate ratio of output to labor inputs measured in current dollars in the base period, the aggregate Paasche volume measure of labor input productivity is:

$$\frac{(V_t/P_t^L)/(B_t/W_t^L)}{V_0/B_0} - 1 = \frac{\hat{z}_t - \hat{z}_0}{\hat{z}_0}$$

$$= \frac{\sum_i b_{it}\hat{z}_{it} - \hat{b}_{i0}\hat{z}_{i0}}{\hat{z}_0}$$

$$= \sum_i \left[0.5(\hat{b}_{i0} + b_{it})(\hat{z}_{it} - \hat{z}_{i0}) + 0.5(\hat{z}_{i0} + \hat{z}_{it})(b_{it} - \hat{b}_{i0})\right]/\hat{z}_0$$

$$= \sum_i 0.5(1+b_{it}/\hat{b}_{i0})g(\hat{z}_i) + \sum_i \left[0.5(\hat{z}_{i0} + \hat{z}_{it})/\hat{z}_0\right](\hat{b}_{it} - b_{i0})$$

$$= \sum_i 0.5(1+b_{it}/\hat{b}_{i0})g(\hat{z}_i) + \sum_i \left[0.5(\hat{z}_{i0} + \hat{z}_{it} - (\hat{z}_0 + \hat{z}_t))/\hat{z}_0\right](\hat{b}_{it} - b_{i0}) \tag{A9}$$

The contribution to aggregate Paasche labor input productivity growth from within-industry labor input productivity growth in industry i is:

$$c_i^{P\text{-}D} = 0.5(1+b_{it}/\hat{b}_{i0})g(\hat{z}_i) \tag{A10}$$

The contribution of reallocation of labor inputs to or from industry i to aggregate Laspeyres labor input productivity growth is therefore:

$$c_i^{P\text{-}R} = \left[0.5(\hat{z}_{i0} + \hat{z}_{it} - (\hat{z}_0 + \hat{z}_t))/\hat{z}_0\right](b_{it} - \hat{b}_{i0}) \tag{A11}$$

Finally, we can use λ from equation (10) to define Fisher index contributions to aggregate labor inputs productivity change. The direct Fisher contribution of within-industry productivity growth is then seen to be:

$$c_i^{F\text{-}D} = \lambda c_i^{L\text{-}D} + (1-\lambda)c_i^{P\text{-}D}$$

$$= \lambda 0.5(1+\hat{b}_{it}/b_{i0})g(\hat{Z}_i) + (1-\lambda)0.5(1+b_{it}/\hat{b}_{i0})g(\hat{z}_i)$$

$$= 0.5\left[1 + \lambda(\hat{b}_{it}/b_{i0})g(\hat{Z}_i) + (1-\lambda)(b_{it}/\hat{b}_{i0})g(\hat{z}_i)\right] \tag{A12}$$

The contribution of reallocation of labor inputs involving industry i to Fisher labor inputs productivity is then:

$$c_i^{F\text{-}R} = \lambda c_i^{L\text{-}R} + (1-\lambda) c_i^{P\text{-}R}$$

$$= 0.5\Big[\lambda[(\hat{Z}_{i0}+\hat{Z}_{it} - (\hat{Z}_0+\hat{Z}_t))/\hat{Z}_0](\hat{b}_{it} - b_{i0}) + (1-\lambda)[(\hat{z}_{i0}+\hat{z}_{it} - (\hat{z}_0+\hat{z}_t))/\hat{z}_0](b_{it} - \hat{b}_{i0})]\Big] \qquad \text{(A13)}$$

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