Appendix 1: Explaining Real Income Growth: The Translog Approach

The Production Theory Framework

In this Appendix, we present the production theory framework used in the article "New Estimates of Real Income and Multifactor Productivity Growth for the Canadian Business Sector, 1961-2011" published in the Fall 2012 issue of the *International Productivity Monitor*.[1] The main reference is Diewert and Morrison (1986), but we also draw on the theory of the output price index, which was developed by Fisher and Shell (1972) and Archibald (1977).[2] This theory is the producer theory counterpart to the theory of the cost of living index for a single consumer (or household) that was first developed by the Russian economist, A. A. Konüs (1924). These economic approaches to price indexes rely on the assumption of (competitive) *optimizing behavior* on the part of economic agents (consumers or producers). Thus we consider only the market sector of the economy in what follows; i.e. that part of the economy that is motivated by profit maximizing behavior. In our empirical work, we define the market sector to be the Canadian business sector of the economy less the rental and owner occupied housing sectors.[3]

Initially, we assume that the market sector of the economy produces quantities of M (net)[4] outputs, $y \equiv [y_1,...,y_M]$, which are sold at the positive producer prices $P \equiv [P_1,...,P_M]$. We further assume that the market sector of the economy uses positive quantities of N primary inputs, $x \equiv [x_1,...,x_N]$ which are purchased at the positive primary input prices $W \equiv [W_1,...,W_N]$. In period t, we assume that there is a feasible set of output vectors y that can be produced by the market sector if the vector of primary inputs x is utilized by the market sector of the economy; denote this period t production possibilities set by S^t. We assume that S^t is a closed convex cone that exhibits a free disposal property.[5]

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[1] This material is drawn from Diewert, Mizobuchi and Nomura (2005) and Diewert and Lawrence (2006).

[2] The theory also draws on Diewert (1983:1077-1100), Kohli (1990, 2003, 2004, and 2006), Morrison and Diewert (1990), Fox and Kohli (1998) and Chapter 24 in the IMF, ILO, OECD, Eurostat, UNECE and the World Bank (2009).

[3] The Canadian business sector excludes all of the general government sectors such as schools, hospitals, universities, defense and public administration where no independent measures of output can be obtained. For owner occupied housing, output is equal to input and hence no productivity improvements can be generated by this sector according to SNA conventions. Due to the difficulties involved in splitting up the residential housing stock into the rental and owner occupied portions, we omit the entire residential housing stock and the consumption of residential housing services in our data. However, we do include investment in residential housing, since that investment is part of the output of the market production sector.

[4] If the mth commodity is an import (or other produced input) into the market sector of the economy, then the corresponding quantity y_m is indexed with a negative sign. We will follow Kohli (1978 and 1991) and Woodland (1982) in assuming that imports flow through the domestic production sector and are "transformed" (perhaps only by adding transportation, wholesaling and retailing margins) by the domestic production sector. The recent textbook by Feenstra (2004: 76) also uses this approach.

[5] For a more explanation for the meaning of these properties, see Diewert (1974: 134). The assumption that S^t is a cone means that the technology is subject to constant returns to scale. This is an important assumption since it implies that the value of outputs should equal the value of inputs in equilibrium. In our empirical work, we use an ex post rate of return in our user costs of capital, which forces the value of inputs to equal the value of outputs for each period. The function g^t is known as the *GDP function* or the *national product function* in the international trade

Given a vector of output prices P and a vector of available primary inputs x, we define *the period t market sector GDP function*, $g^t(P,x)$, as follows:[6]

(1) $g^t(P,x) \equiv \max_y \{P\cdot y : (y,x) \text{ belongs to } S^t\}$; $t = 0,1,2, \ldots$.

Thus market sector GDP depends on t (which represents the period t technology set S^t), on the vector of output prices P that the market sector faces and on x, the vector of primary inputs that is available to the market sector.

If P^t is the period t output price vector and x^t is the vector of inputs used by the market sector during period t and if the GDP function is differentiable with respect to the components of P at the point P^t,x^t, then the period t vector of market sector outputs y^t will be equal to the vector of first order partial derivatives of $g^t(P^t,x^t)$ with respect to the components of P; i.e., we will have the following equations for each period t:[7]

(2) $y^t = \nabla_P\, g^t(P^t,x^t)$; $t = 0,1,2, \ldots$.

Thus the period t market sector supply vector y^t can be obtained by differentiating the period t market sector GDP function with respect to the components of the period t output price vector P^t.

If the GDP function is differentiable with respect to the components of x at the point P^t,x^t, then the period t vector of input prices W^t will be equal to the vector of first order partial derivatives of $g^t(P^t,x^t)$ with respect to the components of x; i.e., we will have the following equations for each period t:[8]

(3) $W^t = \nabla_x\, g^t(P^t,x^t)$; $t = 0,1,2, \ldots$.

Thus the period t market sector input prices W^t paid to primary inputs can be obtained by differentiating the period t market sector GDP function with respect to the components of the period t input quantity vector x^t.

The constant returns to scale assumption on the technology sets S^t implies that the value of outputs will equal the value of inputs in period t; i.e., we have the following relationships:

(4) $g^t(P^t,x^t) = P^t\cdot y^t = W^t\cdot x^t$; $t = 0,1,2, \ldots$.

The above material will be useful in what follows but of course, our focus is not on GDP; instead our focus is on the income generated by the market sector or more precisely, on *the real income*

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literature; see Kohli (1978 and 1991), Woodland (1982) and Feenstra (2004:76). It was introduced into the economics literature by Samuelson (1953).

[6] The function $g^t(P,x)$ will be linearly homogeneous and convex in the components of P and linearly homogeneous and concave in the components of x; see Diewert (1974: 136). Notation: $P\cdot y \equiv \sum_{m=1}^{M} P_m y_m$.

[7] These relationships are due to Hotelling (1932: 594). Note that $\nabla_P\, g^t(P^t,x^t) \equiv [\partial g^t(P^t,x^t)/\partial P_1, \ldots,\partial g^t(P^t,x^t)/\partial P_M]$.

[8] These relationships are due to Samuelson (1953) and Diewert (1974: 140). Note that $\nabla x\; g^t(P^t,x^t) \equiv [\partial g^t(P^t,x^t)/\partial x_1, \ldots,\partial g^t(P^t,x^t)/\partial x_N]$.

generated by the market sector. However, since market sector GDP (the value of market sector production) is distributed to the factors of production used by the market sector, nominal market sector GDP will be equal to nominal market sector income; i.e. from (4), we have $g^t(P^t,x^t) = P^t\cdot y^t = W^t\cdot x^t$. As an approximate welfare measure that can be associated with market sector production,[9] we will choose to measure the *real income generated by the market sector in period t*, ρ^t, in terms of the number of consumption bundles that the nominal income could purchase in period t; i.e., define ρ^t as follows:

$$(5)\ \rho^t \equiv W^t\cdot x^t/P_C{}^t \, ; \qquad t = 0,1,2, ...$$
$$= w^t\cdot x^t$$
$$= p^t\cdot y^t$$
$$= g^t(p^t,x^t)$$

where $P_C{}^t > 0$ is the *period t consumption expenditures deflator* and the market sector period t *real output price* p^t and *real input price* w^t vectors are defined as the corresponding nominal price vectors deflated by the consumption expenditures price index; i.e., we have the following definitions:[10]

$$(6)\ p^t \equiv P^t/P_C{}^t \, ; \ w^t \equiv W^t/P_C{}^t \, ; \qquad t = 0,1,2, ...$$

The first and last equality in (5) imply that period t real income, ρ^t, is equal to the period t GDP function, evaluated at the period t real output price vector p^t and the period t input vector x^t, $g^t(p^t,x^t)$. Thus *the growth in real income over time can be explained by three main factors: t (Technical Progress or Total Factor Productivity growth), growth in real output prices and the growth of primary inputs*. We will shortly give formal definitions for these three growth factors.

Using the linear homogeneity properties of the GDP functions $g^t(P,x)$ in P and x separately, we can show that the following counterparts to the relations (2) and (3) hold using the deflated prices p and w:[11]

$$(7)\ y^t = \nabla_p\, g^t(p^t,x^t) \, ; \qquad t = 0,1,2, ... ;$$
$$(8)\ w^t = \nabla_x\, g^t(p^t,x^t) \, ; \qquad t = 0,1,2, ...$$

Now we are ready to define a family of *period t productivity growth factors or technical progress shift factors* $\tau(p,x,t)$:[12]

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[9] Since some of the primary inputs used by the market sector can be owned by foreigners, our measure of *domestic* welfare generated by the market production sector is only an approximate one. Moreover, our suggested welfare measure is not sensitive to the distribution of the income that is generated by the market sector.

[10] Our approach is similar to the approach advocated by Kohli (2004: 92), except he essentially deflates nominal GDP by the domestic expenditures deflator rather than just the domestic (household) expenditures deflator; i.e., he deflates by the deflator for C+G+I, whereas we suggest deflating by the deflator for C. Another difference in his approach compared to the present approach is that we restrict our analysis to the market sector GDP, whereas Kohli deflates all of GDP. Our treatment of the balance of trade surplus or deficit is also different.

[11] If producers in the market sector of the economy are solving the profit maximization problem that is associated with $g^t(P,x)$, which uses the original output prices P, then they will also solve the profit maximization problem that uses the normalized output prices $p \equiv P/P_C$; i.e., they will also solve the problem defined by $g^t(p,x)$.

(9) $\tau(p,x,t) \equiv g^t(p,x)/g^{t-1}(p,x)$; $t = 1,2, \ldots$.

Thus $\tau(p,x,t)$ measures the proportional change in the real income produced by the market sector at the reference real output prices p and reference input quantities used by the market sector x where the numerator in (9) uses the period t technology and the denominator in (9) uses the period t−1 technology. Thus each choice of reference vectors p and x will generate a possibly different measure of the shift in technology going from period t−1 to period t.

It is natural to choose special reference vectors for the measure of technical progress defined by (9): a *Laspeyres type measure* τ_L^t that chooses the period t−1 reference vectors p^{t-1} and x^{t-1} and a *Paasche type measure* τ_P^t that chooses the period t reference vectors p^t and x^t:

(10) $\tau_L^t \equiv \tau(p^{t-1},x^{t-1},t) = g^t(p^{t-1},x^{t-1})/g^{t-1}(p^{t-1},x^{t-1})$; $t = 1,2, \ldots$;
(11) $\tau_P^t \equiv \tau(p^t,x^t,t) \quad = g^t(p^t,x^t)/g^{t-1}(p^t,x^t)$; $t = 1,2, \ldots$.

Since both measures of technical progress are equally valid, it is natural to average them to obtain an overall measure of technical change. If we want to treat the two measures in a symmetric manner and we want the measure to satisfy the time reversal property from index number theory[13] (so that the estimate going backwards is equal to the reciprocal of the estimate going forwards), then the geometric mean will be the best simple average to take.[14] Thus we define the geometric mean of (10) and (11) as follows:[15]

(12) $\tau^t \equiv [\tau_L^t\, \tau_P^t]^{1/2}$; $t = 1,2, \ldots$.

At this point, it is not clear how we will obtain empirical estimates for the theoretical productivity growth indexes defined by (10)-(12). One obvious way would be to assume a functional form for the GDP function $g^t(p,x)$, collect data on output and input prices and quantities for the market sector for a number of years (and for the consumption expenditures deflator), add error terms to equations (7) and (8) and use econometric techniques to estimate the unknown parameters in the assumed functional form. However, econometric techniques are generally not completely straightforward: different econometricians will make different stochastic specifications and will choose different functional forms. Moreover, as the number of outputs and inputs grows, it will be impossible to estimate a flexible functional form. Thus we will suggest methods for implementing measures like (12) in this Appendix that are based on exact index number techniques.

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[12] This measure of technical progress is due to Diewert and Morrison (1986; 662). A special case of it was defined earlier by Diewert (1983: 1063).
[13] See Fisher (1922: 64).
[14] See the discussion in Diewert (1997) on choosing the "best" symmetric average of Laspeyres and Paasche indexes that will lead to the satisfaction of the time reversal test by the resulting average index.
[15] The theoretical productivity change indexes defined by (10)-(12) were first defined by Diewert and Morrison (1986: 662-663) in the nominal GDP context.

We turn now to the problem of defining theoretical indexes for the effects on real income due to changes in real output prices. Define a family of *period t real output price growth factors* $\alpha(p^{t-1},p^t,x,s)$:[16]

$$(13)\ \alpha(p^{t-1},p^t,x,s) \equiv g^s(p^t,x)/g^s(p^{t-1},x)\ ; \qquad s = 1,2, \ldots .$$

Thus $\alpha(p^{t-1},p^t,x,s)$ measures the proportional change in the real income produced by the market sector that is induced by the change in real output prices going from period t−1 to t, using the technology that is available during period s and using the reference input quantities x. Each choice of the reference technology s and the reference input vector x will generate a possibly different measure of the effect on real income of a change in real output prices going from period t−1 to period t.

Again, it is natural to choose special reference vectors for the measures defined by (13): a *Laspeyres type measure* $\alpha_L{}^t$ that chooses the period t−1 reference technology and reference input vector x^{t-1} and a *Paasche type measure* $\alpha_P{}^t$ that chooses the period t reference technology and reference input vector x^t:

$$(14)\ \alpha_L{}^t \equiv \alpha(p^{t-1},p^t,x^{t-1},t-1) = g^{t-1}(p^t,x^{t-1})/g^{t-1}(p^{t-1},x^{t-1})\ ; \qquad t = 1,2, \ldots ;$$
$$(15)\ \alpha_P{}^t \equiv \alpha(p^{t-1},p^t,x^t,t) = g^t(p^t,x^t)/g^t(p^{t-1},x^t)\ ; \qquad t = 1,2, \ldots .$$

Since both measures of real output price change are equally valid, it is natural to average them to obtain an overall measure of the effects on real income of the change in real output prices:[17]

$$(16)\ \alpha^t \equiv [\alpha_L{}^t\,\alpha_P{}^t]^{1/2}\ ; \qquad t = 1,2, \ldots .$$

Finally, we look at the problem of defining theoretical indexes for the effects on real income due to changes in input quantities. Define a family of *period t real input quantity growth factors* $\beta(x^{t-1},x^t,p,s)$:[18]

$$(17)\ \beta(x^{t-1},x^t,p,s) \equiv g^s(p,x^t)/g^s(p,x^{t-1})\ ; \qquad s = 1,2, \ldots .$$

Thus $\beta(x^{t-1},x^t,p,s)$ measures the proportional change in the real income produced by the market sector that is induced by the change in input quantities used by the market sector going from period t−1 to t, using the technology that is available during period s and using the reference real output prices p. Each choice of the reference technology s and the reference real output price

[16] This measure of real output price change was essentially defined by Fisher and Shell (1972: 56-58), Samuelson and Swamy (1974: 588-592), Archibald (1977: 60-61), Diewert (1980: 460-461 and 1983:1055) and Balk (1998: 83-89). Readers who are familiar with the theory of the true cost of living index will note that the real output price index defined by (13) is analogous to the Konüs (1924) *true cost of living index* which is a ratio of cost functions, say $C(u,p^t)/C(u,p^{t-1})$ where u is a reference utility level: g^s replaces C and the reference utility level u is replaced by the vector of reference variables x.

[17] The indexes defined by (13)-(16) were defined by Diewert and Morrison (1986: 664) in the nominal GDP function context.

[18] This type of index was defined as a true index of value added by Sato (1976: 438) and as a real input index by Diewert (1980: 456).

vector p will generate a possibly different measure of the effect on real income of a change in input quantities going from period t−1 to period t.

As usual, it is natural to choose special reference vectors for the measures defined by (17): a *Laspeyres type measure* $\beta_L{}^t$ that chooses the period t−1 reference technology and reference real output price vector p^{t-1} and a *Paasche type measure* $\beta_P{}^t$ that chooses the period t reference technology and reference real output price vector p^t:

$$(18)\ \beta_L{}^t \equiv \beta(x^{t-1},x^t,p^{t-1},t-1) = g^{t-1}(p^{t-1},x^t)/g^{t-1}(p^{t-1},x^{t-1}) ; \qquad t = 1,2, \ldots ;$$
$$(19)\ \beta_P{}^t \equiv \beta(x^{t-1},x^t,p^t,t) = g^t(p^t,x^t)/g^t(p^t,x^{t-1}) ; \qquad t = 1,2, \ldots .$$

Since both measures of real input growth are equally valid, it is natural to average them to obtain an overall measure of the effects of input growth on real income:[19]

$$(20)\ \beta^t \equiv [\beta_L{}^t\, \beta_P{}^t]^{1/2} ; \qquad t = 1,2, \ldots .$$

Recall that market sector real income for period t was defined by (5) as ρ^t equal to nominal period t factor payments $W^t \cdot x^t$ deflated by the household consumption price deflator $P_C{}^t$. It is convenient to define γ^t as the *period t chain rate of growth factor for real income*:

$$(21)\ \gamma^t \equiv \rho^t/\rho^{t-1} ; \qquad t = 1,2, \ldots .$$

The Translog GDP Function Approach

We now follow the example of Diewert and Morrison (1986: 663) and assume that the log of the period t (deflated) GDP function, $g^t(p,x)$, has the following translog functional form:[20]

$$(22)\ \ln g^t(p,x) \equiv a_0{}^t + \sum_{m=1}^M a_m{}^t \ln p_m{}^t + (1/2) \sum_{m=1}^M\sum_{k=1}^M a_{mk} \ln p_m{}^t \ln p_k{}^t$$
$$+ \sum_{n=1}^N b_n{}^t \ln x_n{}^t + (1/2)\sum_{n=1}^N\sum_{j=1}^N b_{nj} \ln x_n{}^t \ln x_j{}^t + \sum_{m=1}^M\sum_{n=1}^M c_{mn} \ln p_m{}^t \ln x_n{}^t ;$$
$$t = 0,1,2, \ldots .$$

Note that the coefficients for the quadratic terms are assumed to be constant over time. The coefficients must satisfy the following restrictions in order for g^t to satisfy the linear homogeneity properties that we have assumed in section 2 above:[21]

$$(23)\ \sum_{m=1}^M a_m{}^t = 1 \text{ for } t = 0,1,2, \ldots;$$
$$(24)\ \sum_{n=1}^N b_n{}^t = 1 \text{ for } t = 0,1,2, \ldots;$$
$$(25)\ a_{mk} = a_{km} \text{ for all } k,m ;$$

[19] The theoretical indexes defined by (17)-(20) were defined in Diewert and Morrison (1986: 665) in the nominal GDP context.

[20] This functional form was first suggested by Diewert (1974: 139) as a generalization of the translog functional form introduced by Christensen, Jorgenson and Lau (1971). Diewert (1974: 139) indicated that this functional form was flexible.

[21] There are additional restrictions on the parameters which are necessary to ensure that $g^t(p,x)$ is convex in p and concave in x. Note that when we divide the original prices by one of the prices, then one of the scaled prices will be identically equal to one and hence its logarithm will be identically equal to zero.

(26) $b_{nj} = b_{jn}$ for all n,j ;
(27) $\sum_{k=1}^{M} a_{mk} = 0$ for m = 1,...,M ;
(28) $\sum_{j=1}^{N} b_{nj} = 0$ for n = 1,...,N ;
(29) $\sum_{n=1}^{N} c_{mn} = 0$ for m = 1,...,M ;
(30) $\sum_{m=1}^{M} c_{mn} = 0$ for n = 1,...,N .

Diewert and Morrison (1986: 663) showed that[22] if g^{t-1} and g^t are defined by (22)-(30) above and there is competitive profit maximizing behavior on the part of all market sector producers for all periods t, then

(31) $\gamma^t = \tau^t\, \alpha^t\, \beta^t$; t = 1,2, ...

where γ^t, τ^t, α^t and β^t are defined above by (21), (12), (16) and (20) respectively. In addition, Diewert and Morrison (1986: 663-665) showed that τ^t, α^t and β^t could be calculated using empirically observable price and quantity data for periods t−1 and t as follows:

(32) $\ln\alpha^t = \sum_{m=1}^{M} (1/2)[(p_m^{t-1}y_m^{t-1}/p^{t-1}\cdot y^{t-1}) + (p_m^{t}y_m^{t}/p^{t}\cdot y^{t})] \ln(p_m^{t}/p_m^{t-1})$
$= \ln P_T(p^{t-1},p^t,y^{t-1},y^t)$;
(33) $\ln\beta^t = \sum_{n=1}^{N} (1/2)[(w_n^{t-1}x_n^{t-1}/w^{t-1}\cdot x^{t-1}) + (w_n^{t}x_n^{t}/w^{t}\cdot x^{t})] \ln(x_n^{t}/x_n^{t-1})$
$= \ln Q_T(w^{t-1},w^t,x^{t-1},x^t)$;
(34) $\tau^t = \gamma^t/\alpha^t\,\beta^t$

where $P_T(p^{t-1},p^t,y^{t-1},y^t)$ is the Törnqvist (1936) and Törnqvist and Törnqvist (1937) output price index and $Q_T(w^{t-1},w^t,x^{t-1},x^t)$ is the Törnqvist input quantity index.

Equations (31) are in rates of growth. It is possible to obtain counterparts to these equations in a levels form as follows. Thus we can express the level of real income in period t in terms of an *index of the technology level* or of Total Factor Productivity in period t T^t, of the *level of real output prices* in period t A^t, and of the *level of primary input quantities* in period t, B^t.[23] Thus we use the growth factors τ^t, α^t and β^t as follows to define the levels T^t, A^t and B^t:

(35) $T^0 \equiv 1$; $T^t \equiv T^{t-1}\,\tau^t$; t = 1,2, ... ;
(36) $A^0 \equiv 1$; $A^t \equiv A^{t-1}\alpha^t$; t = 1,2, ... ;
(37) $B^0 \equiv 1$; $B^t \equiv B^{t-1}\beta^t$; t = 1,2, ...

Using the above definitions and the exact equations (31), we can establish the following exact relationship for the level of real income in period t, ρ^t, and the period t levels for technology, real output prices and input quantities:

(38) $\rho^t/\rho^0 = T^t\, A^t\, B^t$; t = 1,2, ...

[22] Diewert and Morrison established their proof using the nominal GDP function $g^t(P,x)$. However, it is easy to rework their proof using the deflated GDP function $g^t(p,x)$ using the fact that $g^t(p,x) = g^t(P/P_C,x) = g^t(P,x)/P_C$ using the linear homogeneity property of $g^t(P,x)$ in P.
[23] This type of levels presentation of the data is quite instructive when presented in graphical form. It was suggested by Kohli (1990) and used extensively by him; see Kohli (2003 and 2004) and Fox and Kohli (1998).

The Translog GDP Function Approach and Changes in the Terms of Trade

For some purposes, it is convenient to decompose the aggregate period t contribution factor due to changes in all deflated output prices α^t into separate effects for each change in each output price. Similarly, it can sometimes be useful to decompose the aggregate period t contribution factor due to changes in all market sector primary input quantities β^t into separate effects for each change in each input quantity. In this section, we indicate how this can be done, making the same assumptions on the technology that were made in the previous section.

We first model the effects of a change in a single (deflated) output price, say p_m, going from period t−1 to t. Counterparts to the theoretical Laspeyres and Paasche type price indexes defined by (14) and (15) above for changes in all (deflated) output prices are the following *Laspeyres type measure* α_{Lm}^t that chooses the period t−1 reference technology and holds constant other output prices at their period t−1 levels and holds inputs constant at their period t−1 levels x^{t-1} and a *Paasche type measure* α_{Pm}^t that chooses the period t reference technology and reference input vector x^t and holds constant other output prices at their period t levels:

$$(39)\ \alpha_{Lm}^t \equiv g^{t-1}(p_1^{t-1},...,p_{m-1}^{t-1},p_m^t,p_{m+1}^{t-1},..., p_M^{t-1},x^{t-1})/g^{t-1}(p^{t-1},x^{t-1});\ m = 1,...,M;\ t = 1,...;$$
$$(40)\ \alpha_{Pm}^t \equiv g^t(p^t,x^t)/g^t(p_1^t,...,p_{m-1}^t,p_m^{t-1},p_{m+1}^t,..., p_M^t,x^t)\ ;\qquad m = 1,...,M;\ t = 1,2, ...$$

Since both measures of real output price change are equally valid, it is natural to average them to obtain an *overall measure of the effects on real income of the change in the real price of output m*:[24]

$$(41)\ \alpha_m^t \equiv [\alpha_{Lm}^t\, \alpha_{Pm}^t]^{1/2}\ ;\qquad m = 1,...,M\ ;\ t = 1,2, ...$$

Under the assumption that the deflated GDP functions $g^t(p,x)$ have the translog functional forms as defined by (22)-(30) in the previous section, the arguments of Diewert and Morrison (1986: 666) can be adapted to give us the following result:

$$(42)\ \ln\alpha_m^t = (1/2)[(p_m^{t-1}y_m^{t-1}/p^{t-1}\cdot y^{t-1}) + (p_m^t y_m^t/p^t\cdot y^t)]\ \ln(p_m^t/p_m^{t-1})\ ;\qquad m = 1,...,M\ ;\ t = 1,2, ...$$

Note that $\ln\alpha_m^t$ is equal to the mth term in the summation of the terms on the right hand side of (32). This observation means that we have the following exact decomposition of the period t aggregate real output price contribution factor α^t into a product of separate price contribution factors; i.e., we have under present assumptions:

$$(43)\ \alpha^t = \alpha_1^t\alpha_2^t ... \alpha_M^t\ ;\qquad t = 1,2, ...$$

The above decomposition is useful for analyzing how real changes in the price of exports (i.e. a change in the price of exports relative to the price of domestic consumption) and in the price of

[24] The indexes defined by (39)-(41) were defined by Diewert and Morrison (1986: 666) in the nominal GDP function context.

imports impact on the real income generated by the market sector. In the empirical illustration which follows later, we let M equal three. The three net outputs are:

- Domestic sales (C+I+G);
- Exports (X) and
- Imports (M).

Since commodities 1 and 2 are outputs, y_1 and y_2 will be positive but since commodity 3 is an input into the market sector, y_3 will be negative. Hence an increase in the real price of exports will *increase* real income but an increase in the real price of imports will *decrease* the real income generated by the market sector, as is evident by looking at the contribution terms defined by (42) for m = 2 (where $y_m^t > 0$) and for m = 3 (where $y_m^t < 0$).

As mentioned above, it is also useful to have a decomposition of the aggregate contribution of input growth to the growth of real income into separate contributions for each important class of primary input that is used by the market sector. We now model the effects of a change in a single input quantity, say x_n, going from period t−1 to t. Counterparts to the theoretical Laspeyres and Paasche type quantity indexes defined by (18) and (19) above for changes in input n are the following *Laspeyres type measure* β_{Ln}^t that chooses the period t−1 reference technology and holds constant other input quantities at their period t−1 levels and holds real output prices at their period t−1 levels p^{t-1} and a *Paasche type measure* β_{Pn}^t that chooses the period t reference technology and reference real output price vector p^t and holds constant other input quantities at their period t levels:

$$(44)\ \beta_{Ln}^t \equiv g^{t-1}(p^{t-1},x_1^{t-1},...,x_{n-1}^{t-1},x_n^t,x_{n+1}^{t-1},..., x_N^{t-1})/g^{t-1}(p^{t-1},x^{t-1})\ ;\quad n = 1,...,N;\ t = 1,2, ... ;$$

$$(45)\ \beta_{Pn}^t \equiv g^t(p^t,x^t)/g^t(p^t,x_1^t,...,x_{n-1}^t,x_n^{t-1},x_{n+1}^t,..., p_N^t)\ ;\quad n = 1,...,N;\ t = 1,2, ...$$

Since both measures of input change are equally valid, as usual, we average them to obtain *an overall measure of the effects on real income of the change in the quantity of input n*:[25]

$$(46)\ \beta_n^t \equiv [\beta_{Pn}^t\,\beta_{Pn}^t]^{1/2}\ ;\qquad n = 1,...,N\ ;\ t = 1,2, ...$$

Under the assumption that the deflated GDP functions $g^t(p,x)$ have the translog functional forms as defined by (22)-(30) in the previous section, the arguments of Diewert and Morrison (1986; 667) can be adapted to give us the following result:

$$(47)\ \ln\beta_n^t = (1/2)[(w_n^{t-1}x_n^{t-1}/w^{t-1}\cdot x^{t-1}) + (w_n^t x_n^t/w^t\cdot x^t)]\ \ln(x_n^t/x_n^{t-1});\ n = 1,...,N\ ;\ t = 1,2, ...$$

Note that $\ln\beta_n^t$ is equal to the nth term in the summation of the terms on the right hand side of (33). This observation means that we have the following exact decomposition of the period t aggregate input growth contribution factor β^t into a product of separate input quantity contribution factors; i.e., we have under present assumptions:

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[25] The indexes defined by (52)-(54) were defined by Diewert and Morrison (1986: 667) in the nominal GDP function context.

(48) $\beta^t = \beta_1{}^t\beta_2{}^t \ldots \beta_N{}^t$; $t = 1,2, \ldots$.

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