Consistency Issues in the Construction of Annual and Quarterly Productivity Indices

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Abstract

Productivity change is generally measured in index form as ratio of output quantity index over input quantity index. Several statistical agencies publish quarterly as well as annual productivity indices, constructed from what appear to be basically the same sources. This raises the question whether, apart from measurement errors, consistency between quarterly and annual indices can be expected. This article explores, from a theoretical perspective, the options for obtaining consistency between annual and quarterly (or more general: between period and sub-period) measures of productivity change.

Productivity change is generally measured in index form as the ratio of an output quantity index over an input quantity index. The presentation is usually in the form of a percentage change (aka growth rate). Specific measures materialize by selecting the output concept to be used (such as gross output or value added) and the number of inputs to be considered (resulting in single, multiple, or total factor productivity indices) (OECD, 2001; Balk, 2018).

The frequency with which such indices are compiled varies. The 2018 edition of the *OECD Compendium of Productivity Indicators* lists annual data for 44 countries. However, a number of official statistical agencies, as well as some international organisations, publish also quarterly data (Haine and Kanutin, 2008). A well-known example is the US Bureau of Labor Statistics where such data have been published since 1967 (Eldridge, Manser, and Otto, 2008). In most cases it appears that annual and quarterly data are constructed independently from basically the same source materials. This raises the issue of consistency between annual and sub-annual index numbers. However, even when annual and quarterly index numbers are by construction not independent, there are issues for concern.

An interesting example is provided by the quarterly labour productivity series published by the UK Office for National Statistics. The basic building block ap-

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pears to be a quarter-of-current-year-relative-to-previous-quarter productivity index, and quarterly productivity change is this index turned into a percentage. The productivity index for a quarter of the current year relative to some reference year (which is currently 2016) is obtained by chaining the quarter-of-current-year-relative-to-previous-quarter productivity indices (and normalizing to get 2016=100). The productivity index for the current year is then defined as the unweighted arithmetic mean of the productivity indices for its four quarters, after seasonal adjustment of the time series. Annual productivity change is obtained by dividing the productivity index for the current year by the same index for the previous year, and turning this ratio into a percentage.

Though in this setup quarterly productivity change has a definite meaning, the situation is less clear for the concept of annual change. In any case, the functional forms of the annual and quarterly productivity indices are grossly different, and likely to coincide only in exceptional circumstances.

Consistency as discussed in this article is a step beyond the consistency concept that figures in the National Accounts literature. There one is concerned with the requirement that annual (real) GDP must be equal to the sum of quarterly (real) GDP, and annual hours worked must be equal to quarterly hours worked, the realisation of which usually invokes some smoothing algorithm. However, as will be shown in more detail in this article, this kind of consistency does not necessarily imply that annual productivity (change) is equal to a simple mean of quarterly productivity (change). The situation is much more complex.

We are touching here a sort of "open nerve". Though users of productivity statistics are well aware of the fact that, due to the complexity of all the survey and compilation processes, one can hardly expect that an independently compiled annual measure of productivity change is equal to a simple or less simple mean of sub-annual measures, the question of what conceptually is at stake here seems to have been avoided. Though Diewert (2008), in his retrospective survey of the two OECD workshops held in 2005-2006, lists the lack of consistency between quarterly estimates of productivity growth and annual estimates as one of the 12 measurement problems where further research is required, as yet no one has taken up this challenge.

This article explores, from a theoretical perspective, the options for obtaining consistency between annual and quarterly (or more general: between period and subperiod) measures of productivity change. It is thereby assumed that all the necessary data are given, without statistical error. We are thus not talking about approximation errors. The article is necessarily a bit more technical than most articles in this journal. This is a simple consequence of the fact that the use of conventional language tends to obscure important conceptual differences. As the UK example demonstrates, the same term "productivity change" is used for quarterly and annual measures which are functionally completely different.

The article unfolds as follows. The first section considers the simple case of a single-input single-output production unit. In this case one can talk about productivity (level) as output quantity divided by input quantity. Annual output or input quantity is the simple sum of sub-annual quantities. Annual productivity then appears to be a weighted mean of sub-annual productivities. Annual productivity change is simply defined as the ratio of two annual productivities. For the definition of sub-annual productivity change there are several options: 1) compare adjacent sub-periods; 2) compare corresponding sub-periods of adjacent years; 3) compare a sub-period to an earlier year. It will appear that, whatever choice is being made, the relation between annual and sub-annual indices is anything but simple.

Section two generalizes this to a multiple-input multiple-output situation, where input and output prices are fixed. Section three considers the general case. The question is, if it be assumed that price, quantity, and productivity indices satisfy some very fundamental axioms, will it then be possible to obtain consistency between annual and sub-annual indices? The answer appears to be negative. Section four considers the use of sub-period productivity indices as approximations to or forecasts of period indices. Section five concludes.

A Simple Case

Let us for a start consider a single-input single-output production unit through two adjacent periods, called 0 and 1 respectively, of equal length. Each period consists of Q sub-periods, also of equal length. The quantity of output produced during sub-period q of period t will be denoted by y^{tq} $(t = 0, 1; q = 1, ..., Q)$. Likewise, the quantity of input used during sub-period q of period t will be denoted by x^{tq} $(t = 0, 1; q = 1, ..., Q)$. All these quantities are assumed to be strictly positive.

The quantity of output produced during the entire period t is evidently measured as the sum of the sub-period quantities,

$$y^t \equiv \sum_{q=1}^{Q} y^{tq} \quad (t = 0, 1). \qquad (1)$$

Likewise, the quantity of input used during the entire period t is evidently measured by

$$x^t \equiv \sum_{q=1}^{Q} x^{tq} \quad (t = 0, 1). \qquad (2)$$

These two relations are basic for what follows.

Productivity

In the case of a single-input single-output unit one can unambiguously talk about productivity as the quantity of output per unit of input. Hence, the productivity in sub-period q of period t is measured by

$$PROD(tq) \equiv y^{tq}/x^{tq} \quad (t = 0, 1; q = 1, ..., Q), \qquad (3)$$

and the productivity in the entire period t by

$$PROD(t) \equiv y^t/x^t \quad (t = 0, 1). \qquad (4)$$

It is straightforward to check, using expressions (1) and (3), that the productivity of any period can be expressed as a weighted arithmetic average of its sub-period productivities,

$$PROD(t) = \sum_{q=1}^{Q} (x^{tq}/x^{t}) PROD(tq), \quad (5)$$

the weights being input quantity shares. Alternatively, by expressions (2) and (3) the productivity of any period can be expressed as a weighted harmonic average of its sub-period productivities,

$$PROD(t) = \left(\sum_{q=1}^{Q} (y^{tq}/y^{t}) (PROD(tq))^{-1} \right)^{-1}, \quad (6)$$

the weights now being output quantity shares.

It is tempting to ask whether $PROD(t)$ can also be expressed as a geometric mean of the sub-period productivities $PROD(tq)$ $(q = 1, ..., Q)$. The answer appears to be negative. Employing the logarithmic mean[2] one obtains

$$\ln PROD(t) = \sum_{q=1}^{Q} \frac{LM(x^{tq}, y^{tq})}{LM(x^{t}, y^{t})} * \ln PROD(tq), \quad (7)$$

or

$$PROD(t) = \prod_{q=1}^{Q} PROD(tq)^{\phi^{tq}}, \quad (8)$$

where $\phi^{tq} \equiv LM(x^{tq}, y^{tq})/LM(x^{t}, y^{t})$ $(q = 1, ..., Q)$. Put otherwise, the temporal aggregate productivity $PROD(t)$ is a weighted product of the sub-period productivities $PROD(tq)$ $(q = 1, ..., Q)$, where the weights are symmetric in input and output quantities. Note however that, due to the concavity of the function $LM(a, 1)$, the sum of these weights is less than or equal to 1, though the difference is usually small. Thus expression (8) is not a geometric mean.

Productivity change

Productivity change between two (sub-) periods, as measured in ratio form, is naturally defined as the ratio of the productivities of the two (sub-) periods considered. In this way the productivity change between periods 0 and 1 is measured by

$$IPROD(1, 0) \equiv \frac{PROD(1)}{PROD(0)} = \frac{y^{1}/x^{1}}{y^{0}/x^{0}}. \quad (9)$$

When considering sub-periods, there are a number of possibilities. In line with the previous definition one could consider the productivity change between two adjacent sub-periods $q-1$ and q of period t; that is,

$$IPROD(tq, t\,q-1) \equiv \frac{PROD(tq)}{PROD(t\,q-1)} = \frac{y^{tq}/x^{tq}}{y^{t\,q-1}/x^{t\,q-1}} \quad (t = 0, 1; q = 1, ..., Q), \quad (10)$$

where we will use the convention that sub-period 0 of period t is the same as sub-period Q of period $t-1$.

A second possibility is to compare the

2 For any two strictly positive real numbers a and b their logarithmic mean is defined by $LM(a, b) \equiv (a-b)/\ln(a/b)$ when $a \neq b$, and $LM(a, a) \equiv a$. It has the following properties: (1) $\min(a, b) \leq LM(a, b) \leq \max(a, b)$; (2) $LM(a, b)$ is continuous; (3) $LM(\lambda a, \lambda b) = \lambda LM(a, b)$ $(\lambda > 0)$; (4) $LM(a, b) = LM(b, a)$; (5) $(ab)^{1/2} \leq LM(a, b) \leq (a+b)/2$; (6) $LM(a, 1)$ is concave. See Balk (2008, 134-136) for details.

productivity of a certain sub-period to the productivity of the corresponding previous sub-period; that is,

$$IPROD(1q,0q) \equiv \frac{PROD(1q)}{PROD(0q)} = \frac{y^{1q}/x^{1q}}{y^{0q}/x^{0q}}\ (q=1,...,Q). \quad (11)$$

A third possibility is to compare the productivity of a certain sub-period to the productivity of the entire previous period; that is,

$$IPROD(1q,0) \equiv \frac{PROD(1q)}{PROD(0)} = \frac{y^{1q}/x^{1q}}{y^{0}/x^{0}}\ (q=1,...,Q). \quad (12)$$

These three are the most usual modes of comparison.

Relations

The interesting question now is: which relations exist between sub-period productivity indices, of whatever type, and period indices?

Let us first look at the sub-period-to-period type indices. By setting $t = 1$ in expression (5) and dividing both sides by $PROD(0)$, we obtain

$$IPROD(1,0) = \sum_{q=1}^{Q}(x^{1q}/x^{1})IPROD(1q,0); \quad (13)$$

that is, $IPROD(1,0)$ can be written as a weighted mean of $IPROD(1q,0)$ $(q = 1,...,Q)$. The weights are the sub-period input quantity shares of period 1, x^{1q}/x^{1} $(q = 1,...,Q)$. What error do we make by replacing these weights by $1/Q$?

Consider the following modification of the last expression:

$$IPROD(1,0) = \sum_{q=1}^{Q}(1/Q)IPROD(1q,0) + \sum_{q=1}^{Q}(x^{1q}/x^{1} - 1/Q)IPROD(1q,0). \quad (14)$$

The second factor at the right-hand side of this expression can be conceived as the covariance of the input quantity shares x^{1q}/x^{1} and the sub-period productivity indices $IPROD(1q,0)$. If this covariance happens to be equal to 0, then $IPROD(1,0)$ is equal to the unweighted arithmetic mean of $IPROD(1q,0)$ $(q = 1,...,Q)$. This assumption, however, is rather strong and, moreover, concerns the comparison period 1, which is unfortunate from the viewpoint of computation in real time.

Similarly, based on expression (6) we obtain

$$IPROD(1,0) = \left(\sum_{q=1}^{Q}(y^{1q}/y^{1})IPROD(1q,0)^{-1}\right)^{-1} = \left(\sum_{q=1}^{Q}(1/Q)IPROD(1q,0)^{-1} + \sum_{q=1}^{Q}(y^{1q}/y^{1} - 1/Q)IPROD(1q,0)^{-1}\right)^{-1}. \quad (15)$$

The second factor at the right-hand side of this expression can be conceived as the covariance of the output quantity shares y^{1q}/y^{1} and the inverse sub-period productivity indices $1/IPROD(1q,0)$. If this covariance happens to be equal to 0, then $IPROD(1,0)$ is equal to the unweighted harmonic mean of $IPROD(1q,0)$ ($q = 1,...,Q$). This is also a strong assumption.

Finally, based on expression (8) we obtain

$$IPROD(1,0) = \prod_{q=1}^{Q} IPROD(1q,0)^{1/Q} \times \prod_{q=1}^{Q} PROD(1q)^{\phi^{1q}-1/Q}. \quad (16)$$

The first factor at the right-hand side is an unweighted geometric mean. The second factor is not necessarily equal to 1.

The relation between $IPROD(1,0)$ and the sub-period-to-corresponding-subperiod indices $IPROD(1q,0q)$ ($q = 1,...,Q$) is less simple. Again, from expression (5) it appears that

$$IPROD(1,0) = \sum_{q=1}^{Q} \frac{x^{1q}PROD(0q)}{x^{1}PROD(0)} \quad IPROD(1q,0q); \quad (17)$$

that is, $IPROD(1,0)$ can be written as a linear combination of the indices $IPROD(1q,0q)$ ($q = 1,...,Q$). One verifies immediately that the weights $x^{1q}PROD(0q)/x^{1}PROD(0)$ do not add up to 1. Sufficient conditions for these weights to be equal to $1/Q$ are that the sub-period input quantity shares are invariant through time, $x^{1q}/x^{1} = x^{0q}/x^{0}$ ($q = 1,...,Q$), and that all the output quantity shares of period 0 are the same, $y^{0q}/y^{0} = 1/Q$ ($q = 1,...,Q$). From a practical point of view, such conditions are difficult to justify.

Alternatively, from expression (6), it appears that we can write

$$IPROD(1,0) = \left(\sum_{q=1}^{Q} \frac{y^{1q}PROD(0)}{y^{1}PROD(0q)} IPROD(1q,0q)^{-1} \right)^{-1}. \quad (18)$$

Thus, $IPROD(1,0)$ can be written as an harmonic combination of the sub-period indices $IPROD(1q,0q)$ ($q = 1,...,Q$). But note that the weights $y^{1q}PROD(0)/y^{1}PROD(0q)$ also do not add up to 1.

Finally, using expression (8), we obtain

$$IPROD(1,0) = \prod_{q=1}^{Q} \frac{PROD(1q)^{\phi^{1q}}}{PROD(0q)^{\phi^{0q}}}. \quad (19)$$

This expression can be decomposed in a number of ways. Using the period 0 viewpoint, we obtain

$$IPROD(1,0) = \prod_{q=1}^{Q} IPROD(1q,0q)^{\phi^{0q}} \times \prod_{q=1}^{Q} PROD(1q)^{\phi^{1q}-\phi^{0q}}. \quad (20)$$

Using the period 1 viewpoint, we obtain

$$IPROD(1,0) = \prod_{q=1}^{Q} IPROD(1q,0q)^{\phi^{1q}} \times \prod_{q=1}^{Q} PROD(0q)^{\phi^{1q}-\phi^{0q}}. \quad (21)$$

Using the "mean" viewpoint, we obtain

$$IPROD(1,0) = \prod_{q=1}^{Q} IPROD(1q,0q)^{(\phi^{0q}+\phi^{1q})/2} \times \prod_{q=1}^{Q} (PROD(0q)PROD(1q))^{(\phi^{1q}-\phi^{0q})/2} \quad (22)$$

It may be clear that the right-most factors of these three expressions are not necessarily equal to 1.

The adjacent sub-period indices $IPROD(tq, t\ q-1)(t = 0,1; q = 1,...,Q)$ can be related to the sub-period-to-corresponding-sub-period indices by chaining,

$$IPROD(1q,0q) = \prod_{\mu=1}^{q} IPROD(1\mu, 1\ \mu-1) \times \prod_{\mu=q+1}^{12} IPROD(0\mu, 0\ \mu-1)\ (q = 1,...,Q). \quad (23)$$

The right-hand side of expression (23) can then be inserted into expression (17), (18), (20), (21), or (22) to obtain a relation between the period 0 to period 1 productivity index $IPROD(1,0)$ and the adjacent sub-period indices. But this relation does not have a simple form.

The conclusion is that already in the extremely simple case of a single-input single-output unit temporal aggregation of productivity indices proves difficult. It is possible to relate sub-period and period productivity indices to each other, but the resulting expressions are not simple.

A More Realistic Case

Let us now consider a production unit that produces M outputs and uses N inputs. The quantity of output m produced during sub-period q of period t will be denoted by y_m^{tq} $(m = 1,...,M; t = 0,1; q = 1,...,Q)$. Likewise, the quantity of input n used during sub-period m of period t will be denoted by x_n^{tq} $(n = 1,...,N; t = 0,1; q = 1,...,Q)$. It is assumed that in each sub-period at least one input quantity and one output quantity is strictly positive.

The quantity of output m produced during the entire period t is evidently measured by

$$y_m^t \equiv \sum_{q=1}^{Q} y_m^{tq}\ (m = 1,...,M; t = 0,1). \quad (24)$$

Likewise, the quantity of input n used during the entire period t is evidently measured by

$$x_n^t \equiv \sum_{q=1}^{Q} x_n^{tq}\ (n = 1,...,N; t = 0,1). \quad (25)$$

When there are multiple inputs and multiple outputs the concept of productivity (level) is no longer unambiguous. Prices are necessary to aggregate quantities. Thus, suppose we have a set of fixed (strictly positive) output prices $p \equiv (p_1,...,p_M)$ and (strictly positive) input prices $w \equiv (w_1,...,w_N)$. The aggregate output quantity produced during sub-period q of period t is then given by

$$p \cdot y^{tq} = \sum_{m=1}^{M} p_m y_m^{tq} \quad (t = 0, 1; q = 1, ..., Q), \quad (26)$$

where vector notation is used to simplify notation and highlight the analogies to the expressions in the previous section. One could also say that $p{\cdot}y^{tq}$ is the sub-period tq output value expressed in *constant prices*. The aggregate output quantity produced during the entire period t is naturally given by

$$p \cdot y^{t} = \sum_{m=1}^{M} p_m y_m^{t} = \sum_{q=1}^{Q} p \cdot y^{tq} \; (t = 0, 1). \quad (27)$$

Likewise, the aggregate input quantity used during sub-period m of period t is given by

$$w \cdot x^{tq} = \sum_{n=1}^{N} w_n x_n^{tq} \quad (t = 0, 1; q = 1, ..., Q). \quad (28)$$

This is the sub-period tq input value expressed in constant prices. The aggregate input quantity used during the entire period t is also naturally given by

$$w \cdot x^{t} = \sum_{n=1}^{N} w_n x_n^{t} = \sum_{q=1}^{Q} w \cdot x^{tq} \quad (t = 0, 1). \quad (29)$$

Recall that it is assumed that all these values are given, without statistical error.

Conditional on input prices w and output prices p, the productivity (level) in sub-period m of period t is measured by

$$PROD(tq) \equiv p \cdot y^{tq} / w \cdot x^{tq} \quad (t = 0, 1; q = 1, ..., Q), \quad (30)$$

and the productivity (level) in the entire period t by

$$PROD(t) \equiv p \cdot y^{t} / w \cdot x^{t} \; (t = 0, 1). \quad (31)$$

This can be expressed in terms of sub-period productivity levels in three ways, namely

$$PROD(t) = \sum_{q=1}^{Q} (w \cdot x^{tq} / w \cdot x^{t}) PROD(tq), \quad (32)$$

$$PROD(t) = \left(\sum_{q=1}^{Q} (p \cdot y^{tq} / p \cdot y^{t}) (PROD(tq))^{-1} \right)^{-1}, \quad (33)$$

and

$$\ln PROD(t) = \sum_{q=1}^{Q} \frac{LM(w \cdot x^{tq}, p \cdot y^{tq})}{LM(w \cdot x^{t}, p \cdot y^{t})} \ln PROD(tq). \quad (34)$$

The definitions of productivity change between two periods, between two sub-periods, and between a sub-period and a period are straightforward. For instance, productivity change between periods 0 and 1 is measured by

$$IPROD(1, 0) \equiv \frac{PROD(1)}{PROD(0)} = \frac{p \cdot y^{1} / w \cdot x^{1}}{p \cdot y^{0} / w \cdot x^{0}}. \quad (35)$$

It is simple to check that the following relations hold:

$$IPROD(1,0) = \sum_{q=1}^{Q}(w \cdot x^{1q}/w \cdot x^{1})IPROD(1q,0), \quad (36)$$

which generalizes expression (13), and

$$IPROD(1,0) = \sum_{q=1}^{Q}\frac{w \cdot x^{1q}}{w \cdot x^{1}}\frac{PROD(0q)}{PROD(0)}IPROD(1q,0q), \quad (37)$$

which generalizes expression (17). Similarly, generalizations of expressions (15), (16), (18), (20), (21), and (22) can be obtained. Moreover, analogous to the way it was done in the previous section, any sub-period-to-corresponding-sub-period productivity index $IPROD(1q, 0q)$ can be written as a chain of adjacent sub-period indices.

Summarizing, by using a set of fixed input and output prices, any multiple-input multiple-output situation can effectively be reduced to a single-input single-output situation.

The System View

It is clear that the productivity index $IPROD(1, 0)$, as defined by expression (35), can be re-expressed as

$$IPROD(1,0) = \frac{p \cdot y^{1}/p \cdot y^{0}}{w \cdot x^{1}/w \cdot x^{0}}; \quad (38)$$

that is, as the ratio of an output quantity index and an input quantity index. The same holds for the other productivity indices considered in the previous section.

These quantity indices have a specific functional form; they are so-called Lowe indices (Balk, 2008:68). An important disadvantage of a Lowe quantity index is that its dual price index violates a rather fundamental axiom. Consider for instance the output quantity index $p \cdot y^1/p \cdot y^0$. The dual price index is obtained by dividing the quantity index into the value ratio $p^1 \cdot y^1/p^0 \cdot y^0$, where p^t $(t = 0, 1)$ denotes the vector of period t output prices. The result is

$$\frac{p^{1} \cdot y^{1}/p \cdot y^{1}}{p^{0} \cdot y^{0}/p \cdot y^{0}}. \quad (39)$$

It is clear that this price index violates the identity axiom, which requires a price index to deliver the outcome 1 whenever the price vectors of the two periods compared are equal, $p^1 = p^0$. Such a violation is generally considered to be undesirable.

An integrated system of price, quantity, and productivity statistics requires functional forms $P_o(.), P_i(.), Q_o(.), Q_i(.)$, such that

$$p^{1} \cdot y^{1}/p^{0} \cdot y^{0} = P_o(p^{1}, y^{1}, p^{0}, y^{0}) \times Q_o(p^{1}, y^{1}, p^{0}, y^{0}) \quad (40)$$

$$w^{1} \cdot x^{1}/w^{0} \cdot x^{0} = P_i(w^{1}, x^{1}, w^{0}, x^{0}) \times Q_i(w^{1}, x^{1}, w^{0}, x^{0}), \quad (41)$$

and a reasonable number of fundamental axioms (or regularity conditions) for price and quantity indices are satisfied (Balk, 2008 and 2018). Here p^t, w^t $(t = 0, 1)$ denote the vectors of period t output and input prices respectively. Notice that the functional forms used at the output side may or may not be the same as those used at the input side (apart from the dimension of the price and quantity vectors involved).

Given these functional forms the productivity index for period 1 relative to period 0 is defined as

$$IPROD(1,0) \equiv \frac{Q_o(p^1, y^1, p^0, y^0)}{Q_i(w^1, x^1, w^0, x^0)}; \quad (42)$$

that is, output quantity index divided by input quantity index.

Consider now the sub-periods. The relation between period and sub-period quantities was presented in expressions (24) and (25). Let p^{tq} and w^{tq} $(t = 0, 1; q = 1, ..., Q)$ denote the vectors of sub-period output and input prices respectively. The relation between period and sub-period prices is, rather naturally, given by

$$p_m^t \equiv \sum_{q=1}^{Q} p_m^{tq} y_m^{tq} / y_m^t \quad (43)$$
$$(m = 1, ..., M; t = 0, 1)$$
$$w_n^t \equiv \sum_{q=1}^{Q} w_n^{tq} x_n^{tq} / x_n^t \quad (44)$$
$$(n = 1, ..., N; t = 0, 1),$$

and the relation between period and sub-period output and input values is similarly given by

$$p^t \cdot y^t = \sum_{q=1}^{Q} p^{tq} \cdot y^{tq} \; (t = 0, 1) \quad (45)$$

$$w^t \cdot x^t = \sum_{q=1}^{Q} w^{tq} \cdot x^{tq} \; (t = 0, 1). \quad (46)$$

Thus, whereas period quantities are simple sums of sub-period quantities, and the same holds for values, period prices are defined as unit values (given sub-period prices).

Corresponding to expression (42), the productivity index for sub-period $1q$ relative to period 0 is then defined as

$$IPROD(1q, 0) \equiv \frac{Q_o(p^{1q}, y^{1q}, p^0, y^0)}{Q_i(w^{1q}, x^{1q}, w^0, x^0)}$$
$$(q = 1, ..., Q). \quad (47)$$

Can these sub-period indices be related to the period index? The answer is obtained by looking at the so-called *profitability* ratio for period 1 relative to period 0, where profitability is defined as the ratio of output value over input value. On the one hand the profitability ratio can be decomposed as

$$\frac{p^1 \cdot y^1 / p^0 \cdot y^0}{w^1 \cdot x^1 / w^0 \cdot x^0} = \frac{P_o(p^1, y^1, p^0, y^0)}{P_i(w^1, x^1, w^0, x^0)} \frac{Q_o(p^1, y^1, p^0, y^0)}{Q_i(w^1, x^1, w^0, x^0)}. \quad (48)$$

But on the other hand, by temporal disaggregation, one obtains

$$\frac{p^1 \cdot y^1 / p^0 \cdot y^0}{w^1 \cdot x^1 / w^0 \cdot x^0} = \sum_{q=1}^{Q} \frac{w^{1q} \cdot x^{1q}}{w^1 \cdot x^1} \cdot \frac{p^{1q} \cdot y^{1q} / p^0 \cdot y^0}{w^{1q} \cdot x^{1q} / w^0 \cdot x^0}$$
$$= \sum_{q=1}^{Q} \left(\frac{w^{1q} \cdot x^{1q}}{w^1 \cdot x^1} \times \frac{P_o(p^{1q}, y^{1q}, p^0, y^0)}{P_i(w^{1q}, x^{1q}, w^0, x^0)} \frac{Q_o(p^{1q}, y^{1q}, p^0, y^0)}{Q_i(w^{1q}, x^{1q}, w^0, x^0)} \right). \quad (49)$$

By combining the last two expressions, using the definitions in expressions (42) and (47), one obtains

$$IPROD(1,0) = \sum_{q=1}^{Q}\left(\frac{w^{1q}\cdot x^{1q}}{w^{1}\cdot x^{1}}\times \frac{P_o(p^{1q},y^{1q},p^0,y^0)/P_o(p^1,y^1,p^0,y^0)}{P_i(w^{1q},x^{1q},w^0,x^0)/P_i(w^1,x^1,w^0,x^0)}\times IPROD(1q,0)\right). \quad (50)$$

This relation between period-to-period and sub-period-to-period productivity indices is not particularly simple. More importantly, since the productivity index at the left-hand side, $IPROD(1,0)$, is based on the same functional form(s) for the quantity indices as the productivity indices at the right-hand side, $IPROD(1q,0)$, the relation generates restrictions on those functional forms. It turns out that these restrictions are impossible to satisfy, except when $Q_o(.)$ and $Q_i(.)$ exhibit the Lowe functional form; that is, $Q_o(.) = p\cdot y^1/p\cdot y^0$ and $Q_i(.) = w\cdot x^1/w\cdot x^0$ where p and w are fixed output and input prices, respectively. But then the dual $P_o(.)$ and $P_i(.)$ violate the fundamental identity axiom.

Sub-period Productivity Indices as Approximations

Given that a consistent system encompassing period and sub-period productivity indices is impossible, can the latter be used as approximations or forecasts of the former? Can, for instance, the sub-period indices $IPROD(1q,0q)$ for $q = 1,...,Q$ be used as approximations or forecasts of the period index $IPROD(1,0)$?

Consider again the period-to-period profitability ratio; that is, the left-hand side of expression (48). Notice that this ratio can be expressed as

$$\frac{(1/Q)p^1\cdot y^1/p^0\cdot y^0}{(1/Q)w^1\cdot x^1/w^0\cdot x^0}. \quad (51)$$

Let δ^{tq} and ϵ^{tq} $(t = 0,1; q = 1,...,Q)$ be defined by

$$\delta^{tq} \equiv w^{tq}\cdot x^{tq} - w^t\cdot x^t/Q \quad (52)$$

$$\epsilon^{tq} \equiv p^{tq}\cdot y^{tq} - p^t\cdot y^t/Q; \quad (53)$$

that is, δ^{tq} and ϵ^{tq} are deviations of actual sub-period values from average sub-period values.

A first-order Taylor series expansion then delivers

$$\frac{p^{1q}\cdot y^{1q}/p^0\cdot y^0}{w^{1q}\cdot x^{1q}/w^0\cdot x^0} = \frac{(1/Q)p^1\cdot y^1/p^0\cdot y^0}{(1/Q)w^1\cdot x^1/w^0\cdot x^0} + R(\delta^{1q},\epsilon^{1q})\ (q = 1,...,Q), \quad (54)$$

where the remainder term $R(.)$ tends to zero when its arguments tend to zero. Thus, if δ^{1q} and ϵ^{1q} are small random fluctuations around 0, then the sub-period-to-period profitability ratios can be seen as approximations to the period-to-period profitability ratio.

By decomposing both sides of expression (54) and rearranging we obtain

$$IPROD(1q,0) = \left(\frac{P_i(w^{1q},x^{1q},w^0,x^0)}{P_o(p^{1q},y^{1q},p^0,y^0)}\frac{P_o(p^1,y^1,p^0,y^0)}{P_i(w^1,x^1,w^0,x^0)}\times IPROD(1,0)\right) + \frac{P_i(w^{1q},x^{1q},w^0,x^0)}{P_o(p^{1q},y^{1q},p^0,y^0)}R(\delta^{1q},\epsilon^{1q}) \quad (55)$$

$(q = 1,...,Q)$.

The factor in front of $IPROD(1,0)$ can be rewritten as

$$\frac{P_i(w^{1q},x^{1q},w^0,x^0)/P_i(w^1,x^1,w^0,x^0)}{P_o(p^{1q},y^{1q},p^0,y^0)/P_o(p^1,y^1,p^0,y^0)}. \quad (56)$$

The numerator is an index comparing sub-period input prices w^{1q} to period input prices w^1, and the denominator is an index comparing sub-period output prices p^{1q} to period output prices p^1. Thus, somewhat loosely stated, if the seasonality of input and output prices is the same, than any sub-period index $IPROD(1q,0)$ is an unbiased forecaster of $IPROD(1,0)$.

Conclusion

It appears that the goal of full consistency between period and sub-period price, quantity and productivity indices is unattainable. Moreover, as argued in section three, this conclusion is independent of the specific functional forms used for the various indices. This impossibility theorem implies that choices must be made.

The first choice concerns what is to be seen as the most natural accounting period for the production unit considered. In most, if not all, cases this will be a year. Annual price, quantity, and productivity comparisons can be based on indices that satisfy the basic axioms (or regularity conditions) and together form a consistent system.

Given the need for sub-annual productivity information, the second choice concerns the type of index to use. As shown, every choice entails at best an approximate relationship between sub-annual and annual indices. The nature of this approximation should be clearly communicated to the public.

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